Geometry of Immersions, Submersions, Harmonic Maps and of Estimation on Statistical Manifolds

A thesis submitted

in partial fulfillment for the award of the degree of

Doctor of Philosophy

by

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Certificate

This is to certify that the thesis titled ***Geometry of Immersions, Submersions, Harmonic Maps and of Estimation on Statistical Manifolds*** submitted by **Mahesh T V**, to the Indian Institute of Space Science and Technology, Thiruvananthapuram, in partial fulfillment for the award of the degree of **Doctor of Philosophy** is a bona fide record of the original work carried out by him under my supervision. The contents of this thesis, in full or in parts, have not been submitted to any other Institute or University for the award of any degree or diploma.

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Declaration

I declare that this thesis titled ***Geometry of Immersions, Submersions, Harmonic Maps and of Estimation on Statistical Manifolds*** submitted in partial fulfillment for the award of the degree of **Doctor of Philosophy** is a record of the original work carried out by me under the supervision of **Dr. Subrahamanian Moosath K S**, and has not formed the basis for the award of any degree, diploma, associateship, fellowship, or other titles in this or any other Institution or University of higher learning. In keeping with the ethical practice in reporting scientific information, due acknowledgments have been made wherever the findings of others have been cited.

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Mahesh T V

Abstract

Information geometry emerged from the geometric study of a statistical model of probability distributions. A statistical model equipped with a Riemannian metric and a pair of dual affine connections is called a statistical manifold. Various geometric aspects of statistical manifolds were studied by many researchers. The main objective of this thesis is to explore certain geometric properties of statistical manifolds and the geometry of estimation.

In Chapter 2, we discuss the geometry of immersions and statistical manifolds. In Section 2.1, we discuss definitions and basic results related to affine immersions. In Proposition (2.1) detailed proof is given for the result that a simply connected statistical manifold can be realized in $\mathbf{R}^{n+1}$ if and only if it is 1-conformally flat. In Section 2.2, we first discuss about the statistical submanifolds and the fundamental equations associated with it. Then in Theorem (2.4) we prove a necessary and sufficient condition for the inherited statistical manifold structures to be dual to each other. Statistical immersion is defined and in Theorem (2.5) we prove a necessary condition for a statistical manifold to be a statistical hypersurface. Also, we prove its converse in Theorem (2.6). Then, in Theorem (2.10) a necessary and sufficient condition for a statistical immersion into a dually flat statistical manifold of codimension one to be minimal is obtained. Also, in Theorem (2.11) a necessary condition is obtained for minimal statistical immersion of statistical manifolds equipped with α-connections. In Section 2.3, centro-affine immersion into $\mathbf{R}^{n+2}$ and the fundamental equations of it are discussed first. Also, in Proposition (2.3) and in Proposition (2.4) a detailed proof of 1-conformal equivalence and (-1)-conformal equivalence of statistical manifold structures in the case of centroaffine immersions into $\mathbf{R}^{n+2}$ are given, respectively. We define centro-affine immersions of codimension two into a dually flat statistical manifold and in Theorem (2.13) we give a necessary and sufficient condition for the inherited statistical manifold structures to be dual to each other. In Theorem (2.14) we show that the inherited statistical manifold structure is conformally-projectively flat in the case of non-degenerate, centro-affine, equiaffine immersion into a dually flat statistical manifold of codimension two. In Section 2.4, we first discuss the affine fundamental form and relations between curvature tensors for affine immersions of general codimension. Then, we define the transversal volume element map for equiaffine statistical immersion of gen-

eral codimension and certain properties are also proved in Lemma (2.3) and in Proposition (2.6).

In Chapter 3, we discuss the geometry of submersions and statistical manifolds. In Section 3.1, definitions of submersion and semi-Riemannian submersion and certain basic results are given. We summarize the definition and basic results of affine submersions with horizontal distribution in Section 3.2. Also, discuss the theorem by Abe and Hasegawa on geodesics comparison for an affine submersion with horizontal distribution. In Section 3.3, we first introduce the concept of a conformal submersion with horizontal distribution for Riemannian manifolds, which is a generalization of the affine submersion with horizontal distribution. Then, in Theorem (3.3) a necessary condition for the existence of such a map is proved. In Theorem (3.6) a necessary and sufficient condition is obtained for $\pi \circ \sigma$ to be a geodesic of $\mathbf{B}$ when σ is a geodesic of $\mathbf{M}$ for a conformal submersion with horizontal distribution. Then, in Proposition (3.5) we prove a necessary and sufficient condition for the horizontal lift of a geodesic to be geodesic. Also, in corollary (3.3) we give a necessary condition for the connection on $\mathbf{B}$ to be complete when the connection on $\mathbf{M}$ is complete for a conformal submersion with horizontal distribution $\pi : \mathbf{M} \longrightarrow \mathbf{B}$. In Section 3.4, we first discuss the affine submersion with horizontal distribution and statistical manifolds. A statistical structure is obtained on the manifold $\mathbf{B}$ induced by the affine submersion $\pi : \mathbf{M} \longrightarrow \mathbf{B}$ with the horizontal distribution $\mathcal{H}(\mathbf{M}) = \mathcal{V}^{\perp}(\mathbf{M})$. In the case of conformal submersion with horizontal distribution in Theorem (3.9) we prove a necessary and sufficient condition for $(\mathbf{M}, \nabla, g_m)$ to become a statistical manifold. Also, in Proposition (3.7) we prove $\pi : (\mathbf{M}, \nabla) \longrightarrow (\mathbf{B}, \nabla^*)$ is a conformal submersion with horizontal distribution if and only if $\pi : (\mathbf{M}, \overline{\nabla}) \longrightarrow (\mathbf{B}, \overline{\nabla}^*)$ is a conformal submersion with horizontal distribution.

Chapter 4 deals with the statistical structures on tangent bundles, harmonic maps between statistical manifolds and between tangent bundles. In Theorem (4.3) of Section 4.1 we prove a necessary and sufficient condition for $T\mathbf{M}$ to become a statistical manifold with the complete lift connection and the Sasaki lift metric. In Section 4.2, we first give a detailed description of the harmonic map using tension field. In Theorem (4.4) we prove a necessary and sufficient condition for the harmonicity of identity map for conformally-projectively equivalent statistical manifolds. Then, conformal statistical submersion is defined which is a generalization of the statistical submersion and in Theorem (4.5) we prove that harmonicity and conformality cannot coexist. In Section 4.3, certain properties of the differential of the tangent map is given first. For statistical manifolds, in Theorem (4.7) we prove that a smooth map $\phi : \mathbf{M} \longrightarrow \mathbf{B}$ is harmonic with respect to ∇ and ∇^* if and only if it is harmonic with respect to the conjugate connections $\overline{\nabla}$ and $\overline{\nabla^*}$. Then, in Theorem (4.8) given a necessary condition for the harmonicity of the tangent map with respect to the complete lift structure on the tangent bundles. Also, in Proposition (4.8) we prove a necessary and sufficient condition for the tangent map to be a statistical submersion.

In Chapter 5, estimation of parameters in statistical manifolds, exponential family and its submanifolds, estimation of parameters in the curved exponential family and Fisher-Neyman sufficient statistic for parametrized models are discussed. In Section 5.1, short account of the statistical properties of an estimator is given. In Theorem (5.2) of Section 5.2 we show that if all ∇^1-autoparallel proper submanifolds of a ± 1-flat statistical manifold $\mathbf{M}$ are exponential then $\mathbf{M}$ is an exponential family. Also, in Theorem (5.3) we prove that if submanifold of a statistical model is an exponential family, then it is a ∇^1-autoparallel submanifold. In the theory of estimation in curved exponential family we give a short account of Amari's geometric conditions for the consistency and efficiency of an estimator in a curved exponential family using ancilliary manifolds. Then discuss the MLE algorithm for estimating parameters in the curved exponential family obtained by Cheng et al. In Section 5.3 we show that the Fisher-Neyman sufficient statistic is invariant under the isostatistical immersions of statistical manifolds in Theorem (5.5).

Contents

Chapter 1

Introduction

1.1 Introduction and Summary

Information Geometry is the geometric study of a statistical model of probability distributions and its application to various problems in science. Statisticians use statistical models to derive inferences; they use families of probability distributions which form, in most cases, a finite dimensional manifold which in information geometry is known as a statistical manifold. Amari and Nagaoka [1] introduced a conjugate structure or duality structure in information geometry which lead to the development of more applications of information geometry. The notion of dually flat structure was also introduced by Amari and Nagaoka [1] when they studied information geometry on Riemannian spaces. By definition, a statistical structure can be viewed as a generalization of a Riemannian structure containing a Riemannian metric and the Levi-Civita connection. So, it is natural to inquire whether the results in Riemannian geometry still hold in the geometry of statistical manifolds. Various geometric aspects of statistical manifolds were studied by many researchers. In this work, our effort is also to explore certain geometric properties of statistical manifolds and the geometry of estimation.

In Chapter 2, we discuss the geometry of immersions and statistical manifolds. Affine differential geometry assumes a significant importance in the field of geometry. Affine immersions were introduced by Nomizu and Sasaki [2]. Statistical manifold was originally introduced by Lauritzen [3], later Kurose [4] reformulated this from the viewpoint of affine differential geometry. Kurose [5] has given a necessary and sufficient condition for a statistical manifold to be realized by an affine immersion of codimension one. Dillen et al. [6] proved a necessary and sufficient condition for realizing a simply connected statistical manifold in $\mathbb{R}^{n+1}$. In Section 2.1, we discuss definitions and basic results related to affine immersions [2]. A detailed proof is given for the result that a simply connected statistical manifold can be realized in $\mathbb{R}^{n+1}$ if and only if it is 1-conformally flat. Then, discussed the theorems by Kurose [4] on affine immersions and statistical manifolds. In Section 2.2, we first discuss about the statistical submanifolds and the fundamental equations associated with it [7]. Uohashi et al. [8] proved that a 1-conformally flat statistical manifold of dimension $n \geq 2$ can be locally ralized as a submanifold of a flat statistical manifold of dimension $n+1$. We considered manifolds immersed in statistical manifolds and obtained a necessary and sufficient condition for the inherited statistical manifold structures to be dual to each other. Statistical immersion is defined and then proved a necessary condition for a statistical manifold to be a statistical hypersurface. Its converse is also proved [9]. Furuhata [10] has proved a necessary and sufficient condition for minimal immersion of a statistical manifold into $\mathbb{R}^{n+1}$. We prove a necessary and sufficient condition for a statistical immersion into a dually flat statistical manifold of codimension one to be minimal. Also, a necessary condition is obtained for minimal statistical immersion of statistical manifolds equipped with α-connections [11]. In Section 2.3, centro-affine immersion into $\mathbb{R}^{n+2}$ and the fundamental equations of it are discussed first [12], [13]. Then, dependence on the change of transversal vector field and change in an immersion are discussed. Also, proof of 1-conformal equivalence and (-1)-conformal equivalence of statistical manifold structures in the case of centro-affine immersions into $\mathbb{R}^{n+2}$ are given in detail [13]. Matsuzoe [14] obtained conditions for a statistical manifold to be realized in $\mathbb{R}^{n+2}$ by centro-affine, equiaffine immersions of codimension two. We define centro-affine immersions of codimension two into a dually flat statistical manifold and give a necessary and sufficient condition for the inherited statistical manifold structures to be dual to each other. Then, show that the inherited statistical manifold structure is conformally-projectively flat in the case of non-degenerate, centro-affine, equiaffine immersion into a dually flat statistical manifold of codimension two. In Section 2.4, we first discuss the affine fundamental form and relations between curvature tensors for affine immersions of general codimension [2]. Equiaffine immersion of general codimension and the transversal volume element map were studied by Koike and Takekuma [15]. We define the transversal volume element map for equiaffine statistical immersion of general codimension and prove certain properties. Then discuss the result by Matsuzoe et al. [16] on sufficient condition for a statistical submanifold of a dually flat statistical manifold to be equiaffine.

In Chapter 3, we discuss the geometry of submersions and statistical manifolds. Riemannian submersion is a special tool in differential geometry and it has got various applications. Notion of submersion is dual to the notion of an immersion. From a statistical viewpoint submersions were first mentioned by Barndroff-Neilsen and Jupp [17]. O'Neill

[18] defined a Riemannian submersion and obtained the fundamental equations of Riemannian submersions for Riemannian manifolds. Also in [19], O'Neill defined a semi-Riemannian submersion. In Section 3.1, definitions of submersion and semi-Riemannian submersion and certain basic results are given. Comparison of geodesics by O'Neill for a semi-Riemannian submersion is also discussed [18], [20]. We summarize the definition and basic results of affine submersions with horizontal distribution in Section 3.2. Also, discuss the theorem by Abe and Hasegawa [21] on geodesics comparison for an affine submersion with horizontal distribution. In Section 3.3, we first introduce the concept of a conformal submersion with horizontal distribution for Riemannian manifolds, which is a generalization of the affine submersion with horizontal distribution [22]. Then, a necessary condition for the existence of such a map is proved. A necessary and sufficient condition is obtained for $\pi \circ \sigma$ to be a geodesic of $\mathbf{B}$ when σ is a geodesic of $\mathbf{M}$ for a conformal submersion with horizontal distribution. Then, prove a necessary and sufficient condition for the horizontal lift of a geodesic to be geodesic. Also, we give a necessary condition for the connection on $\mathbf{B}$ to be complete when the connection on $\mathbf{M}$ is complete for a conformal submersion with horizontal distribution $\pi : \mathbf{M} \longrightarrow \mathbf{B}$. In Section 3.4, we first discuss the affine submersion with horizontal distribution and statistical manifolds. A statistical structure is obtained on the manifold $\mathbf{B}$ induced by the affine submersion $\pi : \mathbf{M} \longrightarrow \mathbf{B}$ with the horizontal distribution $\mathcal{H}(\mathbf{M}) = \mathcal{V}^{\perp}(\mathbf{M})$. Abe and Hasegawa [21] obtained a necessary and sufficient condition for $(\mathbf{M}, \nabla, g_m)$ to become a statistical manifold for an affine submersion with horizontal distribution $\pi : (\mathbf{M}, \nabla) \longrightarrow (\mathbf{B}, \nabla^*)$. In the case of conformal submersion with horizontal distribution we obtained a necessary and sufficient condition for $(\mathbf{M}, \nabla, g_m)$ to become a statistical manifold. Also, we prove $\pi : (\mathbf{M}, \nabla) \longrightarrow (\mathbf{B}, \nabla^*)$ is a conformal submersion with horizontal distribution if and only if $\pi : (\mathbf{M}, \overline{\nabla}) \longrightarrow (\mathbf{B}, \overline{\nabla}^*)$ is a conformal submersion with horizontal distribution [22].

Chapter 4 deals with the statistical structures on tangent bundles, harmonic maps between statistical manifolds and between tangent bundles. In Section 4.1, we discuss the work of Matsuzoe and Inoguchi [23] and Balan et al. [24] on obtaining the various statistical manifold structures on the tangent bundle $T\mathbf{M}$. Then, we prove a necessary and sufficient condition for $T\mathbf{M}$ to become a statistical manifold with the complete lift connection and the Sasaki lift metric [22]. The motivation to study harmonic maps comes from the applications of Riemannian submersion in theoretical physics [25]. Presently, we see an increasing interest in harmonic maps between statistical manifolds [26], [27]. In Section 4.2, we first give a detailed description of the harmonic map using tension field. In [26], Uohashi obtained a condition for the harmonicity on α-conformally equivalent statistical manifolds. We prove a necessary and sufficient condition for the harmonicity of identity map for conformally-projectively equivalent statistical manifolds. Then, conformal statistical submersion is defined which is a generalization of the statistical submersion and prove that harmonicity and conformality cannot coexist [28]. Harmonicity of the tangent maps of tangent bundles endowed with the Sasaki lift metric were studied in [29], [30] for Riemannian manifolds. In [30], Oproiu obtained conditions for the tangent map to be harmonic in the case of tangent bundles equipped with the metrics obtained from the complete lift of metrics and the vertical lift of appropriate tensor fields. In Section 4.3, certain properties of the differential of the tangent map is given first. For statistical manifolds, we prove that a smooth map $\phi : \mathbf{M} \longrightarrow \mathbf{B}$ is harmonic with respect to ∇ and ∇^* if and only if it is harmonic with respect to the conjugate connections $\overline{\nabla}$ and $\overline{\nabla^*}$. Then, given a necessary condition for the harmonicity of the tangent map with respect to the complete lift structure on the tangent bundles. Also, prove a necessary and sufficient condition for the tangent map to be a statistical submersion.

In Chapter 5, estimation of parameters in statistical manifolds, exponential family and its submanifolds, estimation of parameters in the curved exponential family and Fisher-Neyman sufficient statistic for parametrized models are discussed. In [31], Amari discussed the statistical properties of an estimator in a statistical manifold. In Section 5.1, a short account of the statistical properties of an estimator is given. Amari and Nagaoka [1] obtained a necessary and sufficient condition for a submanifold of an exponential family to be exponential. In Section 5.2, we show that if all ∇^1-autoparallel proper submanifolds of a ± 1-flat statistical manifold $\mathbf{M}$ are exponential then $\mathbf{M}$ is an exponential family. Also, we show that if submanifold of a statistical model is an exponential family, then it is a ∇^1-autoparallel submanifold [32]. In the theory of estimation in curved exponential family we discuss Amari's geometric conditions for the consistency and efficiency of an estimator in a curved exponential family using ancilliary manifolds [33], [31]. Then given the MLE algorithm for estimating parameters in the curved exponential family obtained by Cheng et al. [34]. A statistical model is a family $\mathbf{M}$ of probability measures on a measurable space Ω and the sample space can be finite or infinite. For the case of a finite dimensional sample space the theory of statistical manifold structure with the dual connections is well understood. Also a statistical manifold- a Riemannian manifold with each of whose points is a probability distribution - can be embedded into the space of probability measures on a finite set. Infinite dimensional families of probability distributions were first considered by Pistone and Sempi [35]. To deal with infinite dimensional spaces of probability measures Ay et al. [36] developed a functional analytic framework. They introduced the notion of parametrized measure models and obtained the analogue of the structures considered in the finite dimensional information geometry. In [36], Ay et al. have given a necessary and sufficient condition for a statistic to be a Fisher-Neyman sufficient statistic. We show that the Fisher-Neyman sufficient statistic is invariant under the isostatistical immersions of statistical manifolds in Section 5.3.

1.2 Preliminaries

In this section, certain basic concepts regarding differentiable manifolds and statistical manifolds are discussed, [37], [1].

Definition 1.1. A second countable Hausdorff topological space $\mathbf{M}$ is called an n-dimensional topological manifold if it is locally Euclidean. That is, for every point $p \in \mathbf{M}$ there exists an open set $U \subset \mathbf{M}$ containing p and a homeomorphism $\varphi : U \longrightarrow W$, where W is an open subset of $\mathbb{R}^n$.

(U, φ) is called a **coordinate chart** on $\mathbf{M}$ around p and $\varphi = (x^i), i = 1, \cdots, n$ are called **local coordinates** on U. The coordinate chart (U, φ) is called a global chart when $U = \mathbf{M}$ and in that case we have a global coordinate system.

Let (U, φ) and (V, ψ) be two charts on $\mathbf{M}$ such that $U \cap V \neq \emptyset$, the composite map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \longrightarrow \psi(U \cap V)$ is called the transition map. Two charts (U, φ) and (V, ψ) are said to be smoothly compatible if either $U \cap V = \emptyset$ or the transition map $\psi \circ \varphi^{-1}$ is a diffeomorphism.

A collection $\mathcal{A}$ of charts is said to be an atlas for $\mathbf{M}$ if its domains cover $\mathbf{M}$. An atlas $\mathcal{A}$ is said to be smooth if any two charts in $\mathcal{A}$ are smoothly compatible with each other, and $\mathcal{A}$ is a maximal atlas if any chart that is smoothly compatible with every charts in $\mathcal{A}$ is in $\mathcal{A}$. A smooth structure on any topological manifold is a maximal smooth atlas on $\mathbf{M}$. A smooth manifold is a pair $(\mathbf{M}, \mathcal{A})$, where $\mathbf{M}$ is a topological manifold and $\mathcal{A}$ is a smooth structure on $\mathbf{M}$. If $\mathbf{M}$ is a smooth manifold, a chart contained in the given maximal smooth atlas is called a smooth chart.

Definition 1.2. Let $\mathbf{M}$ and $\mathbf{B}$ be smooth manifolds, a map $f : \mathbf{M} \longrightarrow \mathbf{B}$ is said to be a smooth map if for any smooth charts (U, φ) on $\mathbf{M}$ and (V, ψ) on $\mathbf{B}$ the composite map $\psi \circ f \circ \varphi^{-1}$ is smooth from $\varphi(U \cap f^{-1}(V))$ to $\psi(V)$.

If $\mathbf{B} = \mathbb{R}$, then we call f a smooth function on $\mathbf{M}$. Note that in this case $\psi = Id$. The collection of all smooth functions from $\mathbf{M}$ to $\mathbb{R}$ is denoted by $C^\infty(\mathbf{M})$ which is a vector space over $\mathbb{R}$.

Definition 1.3. Let $\mathbf{M}$ be a smooth manifold and let p be a point in $\mathbf{M}$. A linear map $X : C^\infty(\mathbf{M}) \longrightarrow \mathbb{R}$ is called a derivation at p if it satisfies

$$X(g_1 g_2) = g_1(p)X(g_2) + g_2(p)X(g_1),$$

for $g_1, g_2 \in C^\infty(\mathbf{M})$.

The collection of all derivations of $C^\infty(\mathbf{M})$ at p is called the tangent space to $\mathbf{M}$ at p and is denoted by $T_p\mathbf{M}$. An element of $T_p\mathbf{M}$ is called a tangent vector at p.

Let $(U, \varphi = (x^i))$ be a smooth chart on $\mathbf{M}$ around p. Then $\{\frac{\partial}{\partial x^i}|p, i = 1, \cdots, n\}$ form a basis for $T_p\mathbf{M}$. Let $T_p^*\mathbf{M}$ denote the dual space of $T_p\mathbf{M}$ which is also an n-dimensional vector space and $\{dx^i|p, i = 1, \cdots, n\}$ form a basis, where $dx^i|p$ is the differential of x^i at p. Elements of $T_p^*\mathbf{M}$ are called the cotangent vectors at p.

The tangent bundle on $\mathbf{M}$ is denoted by $T\mathbf{M}$ which is the disjoint union of tangent spaces at all points in $\mathbf{M}$. That is,

$$T\mathbf{M} = \bigcup_{p \in \mathbf{M}} T_p\mathbf{M}.$$

Similarly, the cotangent bundle is

$$T^*\mathbf{M} = \bigcup_{p \in \mathbf{M}} T_p^*\mathbf{M}.$$

Definition 1.4. A vector field X on a smooth manifold $\mathbf{M}$ is a map $X : \mathbf{M} \longrightarrow T\mathbf{M}$, which associate each point $p \in \mathbf{M}$ a tangent vector $X_p \in T_p\mathbf{M}$. Vector field X is said to be a smooth vector field if it is smooth as a map. The set of all smooth vector fields on $\mathbf{M}$ is denoted by $\mathcal{X}(\mathbf{M})$.

Let $\mathbf{M}$ be a smooth manifold of dimension n. A **covariant k-tensor** on $\mathbf{M}$ is a multilinear map

$$F : \underbrace{T_p\mathbf{M} \times \cdots \times T_p\mathbf{M}}_{k \text{ copies}} \longrightarrow \mathbb{R}.$$

Similarly, a **contravariant ℓ-tensor** is a multilinear map

$$F : \underbrace{T_p^*\mathbf{M} \times \cdots \times T_p^*\mathbf{M}}_{\ell \text{ copies}} \longrightarrow \mathbb{R}.$$

Also we have tensors of mixed types, a **tensor of type** (ℓ, k) is a multilinear map

$$F : \underbrace{T_p^*\mathbf{M} \times \cdots \times T_p^*\mathbf{M}}_{\ell \text{ copies}} \times \underbrace{T_p\mathbf{M} \times \cdots \times T_p\mathbf{M}}_{k \text{ copies}} \longrightarrow \mathbb{R}.$$

The collection of all tensors of type (ℓ, k) is denoted by $T_\ell^k(T_p\mathbf{M})$, we often denote collection of all covariant k tensors by $T_0^k(T_p\mathbf{M})$ and contravariant ℓ tensors by $T_\ell^0(T_p\mathbf{M})$. The bundle of (ℓ, k)-tensors on $\mathbf{M}$ is denoted by $T_\ell^k\mathbf{M}$, defined as the disjoint union

$$T_\ell^k\mathbf{M} = \bigcup_{p\in\mathbf{M}} T_\ell^k(T_p\mathbf{M}).$$

A **tensor field of type** (ℓ, k) on $\mathbf{M}$ is a smooth section of the tensor bundle $T_\ell^k\mathbf{M}$.

Definition 1.5. A Riemannian metric g on a smooth manifold $\mathbf{M}$ is a tensor field of type $(0, 2)$, such that

1. $g(X, Y) = g(Y, X)$, for $X, Y \in \mathcal{X}(\mathbf{M})$ (Symmetric).
2. $g(X, X) > 0$ if $X \neq 0$ (Positive definite).

Note that the Riemannian metric determines an inner product on each tangent space $T_p\mathbf{M}$. A **Riemannian Manifold** is a smooth manifold equipped with a Riemannian metric.

Remark 1.1. A semi-Riemannian metric g on $\mathbf{M}$ is a map $p \longrightarrow g_p$, where g_p is the non-degenerate, symmetric inner product on $T_p(\mathbf{M})$ and this map is smooth in the sense that for $X, Y \in \mathcal{X}(\mathbf{M})$, $p \longrightarrow g_p(X_p, Y_p)$ is a smooth map on $\mathbf{M}$. A smooth manifold $\mathbf{M}$ together with a semi-Riemannian metric g is called a **semi-Riemannian manifold**.

Let $f : \mathbf{M} \longrightarrow \mathbf{B}$ be a smooth map, for each point $p \in \mathbf{M}$ define a map $f_* : T_p\mathbf{M} \longrightarrow T_{f(p)}\mathbf{B}$, called the **push-forward** of f, by

$$(f_*X_p)(g) = X_p(g \circ f), \quad \text{for} \quad g \in C^\infty(\mathbf{B}).$$

Note that f_*X_p is a derivation at $f(p)$. Let F be a covariant tensor of type $(0, m)$ on $\mathbf{B}$ then define the **pullback** $f^*(F)$ of F under f as follows:

$$f^*(F)(X_1, \cdots X_m)(p) = F(f_*(X_1)_p, \cdots f_*(X_m)_p),$$

for $X_i \in \mathcal{X}(\mathbf{M})$, $p \in \mathbf{M}$ and m is the dimension of $\mathbf{B}$.

Definition 1.6. Let M be a smooth manifold. An affine connection on M, denoted by ∇, is a map from $\mathcal{X}(\mathrm{M}) \times \mathcal{X}(\mathrm{M})$ into $\mathcal{X}(\mathrm{M})$ satisfying the following properties

(1) $\nabla_{X+Y} Z = \nabla_X Z + \nabla_Y Z$,

(2) $\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z$,

(3) $\nabla_{fX} Y = f \nabla_X Y$,

(4) $\nabla_X (fY) = f \nabla_X Y + X(f) Y$,

for $X, Y \in \mathcal{X}(\mathrm{M})$ and $f \in C^\infty(\mathrm{M})$. The torsion of the affine connection ∇ is defined as $\mathcal{T}(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$, for $X, Y \in \mathcal{X}(\mathrm{M})$. The connection ∇ is called torsion-free or symmetric if $\mathcal{T} \equiv 0$.

Let $(U, \varphi = (x^i))$ be a smooth chart in M. Then $\{\partial_i = \frac{\partial}{\partial x^i}, i = 1, \cdots, n\}$ are smooth vector fields on U called the coordinate vector fields. The affine connection ∇ on M can be locally determined by n^3 functions Γ_{ij}^k given by

$$\nabla_{\partial_i} \partial_j = \sum_k \Gamma_{ij}^k \partial_k. \tag{1.1}$$

Γ_{ij}^k are called the **Christoffel symbols** of the affine connection ∇. The Christoffel symbols are also written as $\Gamma_{ijk} = \sum_h \Gamma_{ij}^h g_{kh} = \langle \nabla_{\partial_i} \partial_j, \partial_h \rangle$, these n^3 functions are called the components of ∇.

Definition 1.7. Let M be a Riemannian manifold with Riemannian metric g. An affine connection ∇ is said to be compatible with g if it satisfies the following product rule for $X, Y, Z \in \mathcal{X}(\mathrm{M})$

$$\nabla_X g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z). \tag{1.2}$$

A compatible with g torsion-free affine connection is called the **Levi-Civita connection** for g.

Note. Note that for a given metric g there exists the unique Levi-Civita connection ∇, for it we have

$$\Gamma_{ijk} = \frac{1}{2} (\partial_i g_{jk} + \partial_j g_{ki} + \partial_k g_{ij}),$$

where $g_{ij} = g(\partial_i, \partial_j)$.

A connection on M is said to be flat if $\nabla_{\partial_i} \partial_j = 0$ for some coordinate system (x^i) on M. In that case (x^i) is called an affine coordinate system for M.

Definition 1.8. Let $(\mathbf{M}, g)$ be a Riemannian manifold, a smooth curve $\sigma : [a, b] \longrightarrow \mathbf{M}$ is said to be a geodesic in $\mathbf{M}$ if $\nabla_{\dot{\sigma}}\dot{\sigma} = 0$, where $\dot{\sigma}$ is the tangent vector field.

Next we discuss certain basic concepts regarding statistical manifolds [1], [3], [36], [38], [39]

Let Ω be the sample space associated with some random experiment and $\mathcal{B}$ be the σ-field of subsets of Ω. Then a probability measure P on $(\Omega, \mathcal{B})$ is a measure satisfying $P(\Omega) = 1$ and $(\Omega, \mathcal{B}, P)$ is called a probability space. Consider a family $\mathbf{M}$ of probability distributions on Ω. Suppose each element of $\mathbf{M}$ can be parametrized using n real valued variables $(\theta^1, \cdots, \theta^n)$, that is

$$\mathbf{M} = \{p(x, \theta) : \theta = (\theta^1, \cdots, \theta^n) \in \Theta\},$$

where Θ is an open subset of $\mathbb{R}^n$ and the map $\theta \longrightarrow p(x, \theta)$ is injective. Such a family $\mathbf{M}$ is called an n-dimensional statistical model or a parametric model or simply a model on Ω. Also, we write it as $\mathbf{M} = \{p_\theta\}$. Note that these are finite dimensional parametrized families of measures and in this case the theory of statistical manifold structure with the dual connections is well studied. Infinite dimensional families of probability measures were first considered by Pistone and Sempi [35]. To deal with the infinite dimensional spaces of probability measures Ay et al. [36] developed a functional analytic framework.

We now state some regularity conditions in the case of statistical models (finite dimensional) which are required for the geometric theory [31], [1].

Regularity Conditions

- Θ is an open subset of $\mathbb{R}^n$ and for each $x \in \Omega$, the map $\theta \longrightarrow p(x, \theta)$ is a smooth map.
- Let $\ell(x, \theta) = \log p(x; \theta)$ and $\partial_i = \frac{\partial}{\partial \theta^i}$. Then the n functions $\{\partial_i \ell(x, \theta), i = 1, \cdots, n\}$ are linearly independent. These functions are knowns as **scores**.
- Assume that the order of integration and differentiation may be freely rearranged.
- The moments of scores exists upto necessary orders.
- Assume that the support of p_θ, supp$(p_\theta) = \{x \in \Omega : p(x, \theta) > 0\}$, does not vary with respect to θ. In that case Ω can be redefined to be supp(p_θ). This is equivalent to that $p(x, \theta) > 0$ for all $\theta \in \Theta$ and all $x \in \Omega$.

Definition 1.9. Let $\mathbf{M} = \{p(x, \theta) : \theta \in \Theta \subseteq \mathbb{R}^n\}$ be a statistical model, the mapping $\varphi : \mathbf{M} \longrightarrow \mathbb{R}^n$ defined by $\varphi(p(x, \theta)) = \theta$ allows us to consider $\varphi = [\theta^i]$ as a coordinate system for $\mathbf{M}$. Suppose we have a C^∞ diffeomorphism ψ from Θ onto $\psi(\Theta) \subseteq \mathbb{R}^n$. Then by using $\rho = \psi(\theta)$ as the parameter instead of θ we get $\mathbf{M} = \{p(x, \psi^{-1}(\rho)) \;\; : \;\; \rho \in \psi(\Theta)\}$. This expresses the same family of probability distributions $\mathbf{M} = \{p(x, \theta)\}$. Then $\mathbf{M}$ is a smooth manifold by considering parametrizations which are C^∞ diffeomorphic to each other to be equivalent and is called a statistical manifold. Note that (θ^i) is a global coordinate system on $\mathbf{M}$.

For the statistical manifold $\mathbf{M} = \{p(x, \theta)\}$ define $\ell(x, \theta) = \log p(x, \theta)$ and consider the partial derivatives $\partial_i \ell$ for $i = 1, \cdots, n$. By regularity condition, $\{\partial_i \ell, i = 1, \cdots, n\}$ are linearly independent functions in x. We can construct following n-dimensional vector space spanned by $\partial_i \ell$, for $i = 1, \cdots, n$ as

$$T_\theta^1(\mathbf{M}) = \{A(x) : A(x) = \sum_{i=1}^{n} A^i \partial_i \ell\}$$

Define the expectation with respect to the distribution $p(x, \theta)$ as

$$E_\theta[f] = \int_\Omega f(x) p(x, \theta) dx.$$

Note that $E_\theta[\partial_i \ell] = 0$ since $p(x, \theta)$ satisfies

$$\int_\Omega p(x, \theta) dx = 1.$$

Hence for any random variable $A(x) \in T_\theta^1(\mathbf{M})$ we have $E_\theta[A(x)] = 0$. This expectation induces an inner product on $\mathbf{M}$ in a natural way

$$\langle A(x), B(x) \rangle_\theta = E_\theta[A(x)B(x)]; \quad \text{for} \quad A(x), B(x) \in T_\theta^1(\mathbf{M}).$$

Then for basic vectors ∂_i and ∂_j we have

$$\begin{aligned} g_{ij}(\theta) = \langle \partial_i, \partial_j \rangle_\theta &= E_\theta[\partial_i \ell \partial_j \ell] \\ &= \int_\Omega \partial_i \ell(x; \theta) \partial_j \ell(x; \theta) p(x, \theta) dx. \end{aligned} \tag{1.3}$$

Note that the matrix $G = (g_{ij}(\theta))$ is symmetric $(ie, g_{ij} = g_{ji})$ and that G is positive definite. Hence $g = \langle . \rangle$ defined in (1.3) is a Riemannian metric on statistical manifold $\mathbf{M}$ called the

Fisher information metric. So statistical manifold is a Riemannian manifold with metric as Fisher information metric.

Example 1.1. Let $\Omega = \mathbb{R}$, $n = 2$, $\theta = (\mu, \sigma)$, $\Theta = \{(\mu, \sigma) : -\infty < \mu < \infty, 0 < \sigma < \infty\}$. The family of normal distributions

$$N(\mu, \sigma) = \{p(x, \theta) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}} : \theta = (\mu, \sigma) \in \mathbb{R}^n\}.$$

This is a 2-dimensional manifold which can be identified with the upper half plane. The log likelihood function is given by

$$\ell(x, \theta) = -\frac{-(x-\mu)^2}{2\sigma^2} - \log\sqrt{2\pi}\sigma.$$

Then, the tangent space $T^1_\theta(\mathbf{M})$ is spanned by $\partial_1 = \frac{\partial}{\partial\mu}$ and $\partial_2 = \frac{\partial}{\partial\sigma}$, here

$$\partial_1 = \frac{(x-\mu)}{\sigma^2}, \quad \partial_2 = -\frac{(x-\mu)}{\sigma^3} - \frac{1}{\sigma}.$$

The Fisher information matrix $G(\theta) = (g_{ij})$ is given by

$$\begin{bmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & \frac{2}{\sigma^2} \end{bmatrix}.$$

Statistical manifolds are abstract generalization of the statistical models. There are three equivalent ways of defining a statistical manifold. One of them is to represent the statistical manifold by $(\mathbf{M}, \nabla, g)$, where $\mathbf{M}$ is a Riemannian manifold, ∇ a connection and g a Riemannian metric with ∇g symmetric [4]. Then dual connections are introduced so that the statistical manifold is represented as a quadruplet $(\mathbf{M}, \nabla, \overline{\nabla}, g)$ [1]. Another way is to define the statistical manifold by the triplet $(\mathbf{M}, g, C)$, where C is the $(0, 3)$-tensor on $(\mathbf{M}, g)$ [3]. The third way of introducing a statistical structure is to deduct the Riemannian metric and the conjugate connections on a Riemannian manifold from a given divergence function [40]. A divergence function is a distance like function (which is not symmetric in general) defined on the manifold [1].

Definition 1.10. A semi-Riemannian manifold $(\mathbf{M}, g)$ with a torsion-free affine connection ∇ is called a statistical manifold if ∇g is symmetric.

For a statistical manifold $(\mathbf{M}, \nabla, g)$ the dual connection $\overline{\nabla}$ is defined by

$$Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \overline{\nabla}_X Z) \tag{1.4}$$

for X, Y and Z in $\mathcal{X}(\mathbf{M})$. If (∇, g) is a statistical structure on $\mathbf{M}$, so is $(\overline{\nabla}, g)$. Then $(\mathbf{M}, \overline{\nabla}, g)$ becomes a statistical manifold called the dual statistical manifold of $(\mathbf{M}, \nabla, g)$.

Let R^{∇} be the curvature tensor of ∇ defined by

$$R^{\nabla}(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

for $X, Y, Z \in \mathcal{X}(\mathbf{M})$. Similarly, $R^{\overline{\nabla}}$ be the curvature tensor of $\overline{\nabla}$. It follows from equation (1.4) that

$$g(R^{\nabla}(X,Y)Z, W) = -g(Z, R^{\overline{\nabla}}(X,Y)W), \tag{1.5}$$

for X, Y, Z and W in $\mathcal{X}(\mathbf{M})$. We say $(\mathbf{M}, \nabla, \overline{\nabla}, g)$ has constant curvature k if

$$R^{\nabla}(X,Y)Z = k\{g(Y,Z)X - g(X,Z)Y\}. \tag{1.6}$$

A statistical manifold with curvature zero is called a flat statistical manifold and in that case $(\mathbf{M}, \nabla, \overline{\nabla}, g)$ is called a dually flat statistical manifold.

Definition 1.11. Two statistical manifolds $(\mathbf{M}, \nabla, g)$ and $(\mathbf{M}, \tilde{\nabla}, \tilde{g})$ are said to be α-conformally equivalent if there exists a real valued function ϕ on $\mathbf{M}$ such that

$$\begin{aligned} \tilde{g}(X,Y) &= e^{\phi} g(X,Y) && (1.7)\\ g(\tilde{\nabla}_X Y, Z) &= g(\nabla_X Y, Z) - \frac{1+\alpha}{2} d\phi(Z) g(X,Y) \\ &\quad + \frac{1-\alpha}{2}\{d\phi(X) g(Y,Z) + d\phi(Y) g(X,Z)\}, && (1.8) \end{aligned}$$

where X, Y and Z in $\mathcal{X}(\mathbf{M})$ and α is a real number.

Note. Two statistical manifolds $(\mathbf{M}, \nabla, g)$ and $(\mathbf{M}, \tilde{\nabla}, \tilde{g})$ are α-conformally equivalent if and only if the dual statistical manifolds $(\mathbf{M}, \overline{\nabla}, g)$ and $(\mathbf{M}, \overline{\tilde{\nabla}}, \tilde{g})$ are $(-\alpha)$-conformally equivalent.

Definition 1.12. Two statistical manifolds $(\mathbf{M}, \nabla, g)$ and $(\mathbf{M}, \tilde{\nabla}, \tilde{g})$ are said to be conformally-projectively equivalent if there exist two real valued functions ϕ and ψ on $\mathbf{M}$ such that

$$\begin{aligned}\tilde{g}(X,Y) &= e^{\phi+\psi}g(X,Y) && (1.9)\\ \tilde{\nabla}_X Y &= \nabla_X Y - g(X,Y)grad_g\psi + d(\phi)(X)Y + d(\phi)(Y)X, && (1.10)\end{aligned}$$

where X, Y and Z in $\mathcal{X}(\mathbf{M})$.

Definition of the statistical manifold given by Lauritzen [3] is

Definition 1.13. Let $(\mathbf{M}, g)$ be a semi-Riemannian manifold, a triplet $(\mathbf{M}, g, C)$, where $C : \mathcal{X}(\mathbf{M}) \times \mathcal{X}(\mathbf{M}) \times \mathcal{X}(\mathbf{M}) \longrightarrow C^\infty(\mathbf{M})$ is a trilnear map, is said to be a statistical manifold if $C(X,Y,Z)$ is totally symmetric for $X, Y, Z \in \mathcal{X}(\mathbf{M})$.

Another way to look at statistical manifold is by considering the divergence functions [40].

Definition 1.14. Let $\mathbf{M}$ be an n-dimensional manifold with coordinate system $\theta = (\theta^1, \cdots, \theta^n) = (\theta^i)$. Let the coordinates of the points p, q be $(\theta^i), (\theta^{i^{'}})$, respectively. A divergence function $D : \mathbf{M} \times \mathbf{M} \longrightarrow \mathbb{R}$ is a smooth function satisfying the following conditions

- $D(p,q) \geq 0$ for $p, q \in \mathbf{M}$ with equality holding if and only if $p = q$.
- $\partial_i \partial_{j'} D(p,q) \mid_{p=q}$ is negative definite, where $\partial_i = \frac{\partial}{\partial \theta^i}$ and $\partial_{j'} = \frac{\partial}{\partial \theta^{j'}}$.

In [40], Eguchi defined the unique Riemannian metric g^D and an affine connection ∇^D using the divergence function as follows

$$\begin{aligned}g^D_{ij} &= \langle \partial_i \partial_j \rangle^D_\theta = -\partial_i \partial_{j'} D(p,q) \mid_{p=q}\\ \Gamma^D_{ijk}(\theta) &= \langle \nabla^D_{\partial_i} \partial_j \partial_k \rangle^D_\theta = -\partial_i \partial_j \partial_{k'} D(p,q) \mid_{p=q}.\end{aligned}$$

Then, $(\mathbf{M}, g^D, \nabla^D)$ is a statistical manifold. The dual structure can be computed using the dual $\overline{D}$ of the divergence function D, where $\overline{D}$ is defined as

$$\overline{D}(p,q) = D(q,p), \quad \text{for} \quad p, q \in \mathbf{M}.$$

Note that $g^{\overline{D}} = g^D$ and $\nabla^{\overline{D}}$ and ∇^D are conjugate with respect to g^D [40]. In [41], Matumoto has proved that every torsion-free dualistic structure is induced from a globally defined divergence.

Chapter 2

Geometry of Immersions and Statistical Manifolds

Affine differential geometry, a branch developing out from the classical differential geometry, assumes a significant importance in the field of geometry and it was introduced in the early 1920's most notably by Blaschke [42]. Affine immersions were introduced by Nomizu and Sasaki [2]. Statistical manifold was originally introduced by Lauritzen [3], later Kurose [4] reformulated this from the viewpoint of affine differential geometry.

Kurose [4] has given a necessary and sufficient condition for a statistical manifold to be realized by an affine immersion of codimension one. Dillen et al. [6] proved a necessary and sufficient condition for realizing a simply connected statistical manifold in $\mathbb{R}^{n+1}$. Matsuzoe [14] obtained conditions for a statistical manifold to be realized by centro-affine, equiaffine immersions of codimension two. Uohashi et al. [8] proved that a 1-conformally flat statistical manifold of dimension $n \geq 2$ can be locally ralized as a submanifold of a flat statistical manifold of dimension $(n+1)$. Furuhata [10] has proved a necessary and sufficient condition for minimal immersion of a statistical manifold into $\mathbb{R}^{n+1}$. In this chapter, we consider affine immersions and immersions into statistical manifolds of codimension one and two. Also, mention about the immersions of general codimension.

In Section 2.1, we give the basic definitions and results of affine immersion along with the proof of simply connected statistical manifold realized in $\mathbb{R}^{n+1}$ if and only if it is 1-conformally flat. Then, discuss the theorems by Kurose [4] on affine immersions and statistical manifolds. In Section 2.2, a necessary and sufficient condition for the inherited statistical manifold structures to be dual to each other in the case of immersions to statistical manifolds is obtained. Then, statistical immersion is defined and prove a necessary condition for a statistical manifold to be a statistical hypersurface. Its converse is also proved [9]. Also, we obtained a necessary and sufficient condition for the affine equivalence of non-degenerate equiaffine immersions of statistical manifolds realized in $\mathbb{R}^{n+1}$. Then, proved a necessary and sufficient condition for a statistical immersion into a dually flat statistical manifold of codimension one to be minimal. Also, a necessary condition is obtained for minimal statistical immersion of statistical manifolds equipped with α-connections [11]. In Section 2.3, we present a detailed proof of 1-conformal equivalence and (-1)-conformal equivalence of statistical manifold structures in the case of centro-affine immersions into $\mathbb{R}^{n+2}$. Then, defined the centro-affine immersions of codimension two into a dually flat statistical manifold and given a necessary and sufficient condition for the inherited statistical manifold structures to be dual to each other. Also, we show that the inherited statistical manifold structure is conformally-projectively flat in the case of non-degenerate, centro-affine, equiaffine immersion into a dually flat statistical manifold of codimension two [9]. In Section 2.4, transversal volume element map is defined for equiaffine statistical immersion of general codimension and certain properties also proved.

2.1 Affine Immersions

In this section, we give the basic definitions and results of an affine immersion along with the proof of the result that a simply connected statistical manifold is realized in $\mathbb{R}^{n+1}$ if and only if it is 1-conformally flat [2], [5]. Then, discussed the theorems by Kurose [4] on affine immersions and statistical manifolds.

Let $\mathbf{M}$ and $\tilde{\mathbf{M}}$ be two smooth manifolds of dimensions n and m ($m \geq n$), respectively with the affine connections ∇ on $\mathbf{M}$ and $\tilde{\nabla}$ on $\tilde{\mathbf{M}}$. Let $k = m - n$.

Definition 2.1. An immersion $f : \mathbf{M} \longrightarrow \tilde{\mathbf{M}}$ is said to be an affine immersion if there exists a k-dimensional smooth distribution $\mathcal{N}$ along f which assigns to every point $p \in \mathbf{M}$ a subspace $\mathcal{N}_p$ of $T_{f(p)}(\tilde{\mathbf{M}})$ such that the following equations hold

$$\begin{aligned} T_{f(p)}(\tilde{\mathbf{M}}) &= f_*(T_p(\mathbf{M})) \bigoplus \mathcal{N}_p, &\quad (2.1)\\ (\tilde{\nabla}_X f_* Y)_p &= (f_*(\nabla_X Y))_p + \alpha(X,Y)_p, &\quad (2.2)\end{aligned}$$

where $\alpha(X,Y)_p \in \mathcal{N}_p$ at each point p in $\mathbf{M}$ and X, Y are in $\mathcal{X}(\mathbf{M})$.

Since the distribution $p \in \mathbf{M} \longrightarrow \mathcal{N}_p$ is smooth, each p in $\mathbf{M}$ has a system of k smooth vector fields $\{\xi_1, \cdots, \xi_k\}$ on a neighborhood U of p such that the span of $\{\xi_1(q), \cdots, \xi_k(q)\}$ equal to $\mathcal{N}_q$ for each $q \in U$. This distribution is considered as a bundle of transversal k-subspaces. If $\tilde{\mathbf{M}}$ is a Riemannian manifold with positive definite Riemannian metric $\tilde{g}$, then for the immersion $f : \mathbf{M} \longrightarrow \tilde{\mathbf{M}}$ we can choose the normal space at each point $p \in \mathbf{M}$, namely,

$$\mathcal{N}_p = \{\xi \in T_{f(x)}(\tilde{\mathbf{M}}) : \tilde{g}(\xi, v) = 0, \quad \text{for all} \quad v \in T_p(\mathbf{M})\}.$$

But, in general there is no natural choice of such transversal subspaces. Also, the map $(X, Y) \in \mathcal{X}(\mathbf{M}) \times \mathcal{X}(\mathbf{M}) \mapsto \alpha(X, Y)$ defines for each point $p \in \mathbf{M}$ a symmetric bilinear map

$$T_p(\mathbf{M}) \times T_p(\mathbf{M}) \longrightarrow \mathcal{N}_p.$$

This map α is called the affine fundamental form.

Example 2.1. *Isometrically immersed hypersurface.* Let $(\mathbf{M}_1, g_1)$ and $(\mathbf{M}_2, g_2)$ be Riemannian manifolds of dimensions n and $(n+1)$, respectively with Levi-Civita connections $\overset{g_1}{\nabla}$ on $\mathbf{M}_1$ and $\overset{g_2}{\nabla}$ on $\mathbf{M}_2$. If $f : (\mathbf{M}_1, g_1) \longrightarrow (\mathbf{M}_2, g_2)$ is an immersion with $g_1 = f^* g_2$, then $f : \mathbf{M}_1 \longrightarrow \mathbf{M}_2$ is an affine immersion with a transversal vector field ξ which is given locally as the unit normal vector field.

Example 2.2. *Graph immersion.* Let $\mathbf{M}$ be an n-dimensional manifold with a flat affine connection ∇ and $\psi : \mathbf{M} \longrightarrow \mathbb{R}^n$ be an affine immersion. Since the dimension of $\mathbf{M}$ and $\mathbb{R}^n$ are equal, locally ψ is a diffeomorphism that preserves affine connections. Consider $\mathbb{R}^n$ as a hyperplane H in $\mathbb{R}^{n+1}$ and let ξ be a parallel vector field transversal to H. For any differentiable function $F : \mathbf{M} \longrightarrow \mathbb{R}$, define $f : \mathbf{M} \longrightarrow \mathbb{R}^{n+1}$ by $f(x) = \psi(x) + F(x)\xi$, for $x \in \mathbf{M}$. We have

$$f_*(Y) = \psi_*(Y) + (dF)(Y)\xi, \quad \text{for } Y \in T_x(\mathbf{M}),$$

so f is an immersion. For X and Y in $\mathcal{X}(\mathbf{M})$,

$$\begin{aligned} \tilde{\nabla}_X f_* Y &= \tilde{\nabla}_X \psi_* Y + \tilde{\nabla}_X (YF\xi) = \psi_*(\nabla_X Y) + (XYF)\xi \\ &= f_*(\nabla_X Y) + (XYF - (\nabla_X Y)F)\xi. \end{aligned}$$

Hence f is an affine immersion with $\alpha(X, Y) = XYF - (\nabla_X Y)F$.

Now onwards in this section we consider the case of codimension $k = 1$.

Definition 2.2. Let $\mathbf{M}$ be an n-dimensional manifold with an affine connection ∇. An immersion $f : \mathbf{M} \longrightarrow \mathbb{R}^{n+1}$ is called an affine immersion of codimension one if there exists a transversal vector field ξ on $\mathbf{M}$ such that

$$T_{f(x)}(\mathbb{R}^{n+1}) = f_*(T_x\mathbf{M}) \bigoplus Span\{\xi_x\}, \tag{2.3}$$

for $x \in \mathbf{M}$ and

$$D_X f_* Y = f_*(\nabla_X Y) + h(X, Y)\xi, \tag{2.4}$$

where D is the standard flat connection on $\mathbb{R}^{n+1}$.
The covariant tensor field h of type $(0, 2)$ is called the second fundamental form.
The affine shape operator S and the transversal connection form τ are given by

$$D_X \xi = -f_*(SX) + \tau(X)\xi. \tag{2.5}$$

Definition 2.3. A torsion-free affine connection ∇ on $\mathbf{M}$ is called equiaffine if there exists a volume element ω on $\mathbf{M}$ such that $\nabla\omega = 0$. Then, we say that (∇, ω) is an equiaffine structure on $\mathbf{M}$.

Consider an affine immersion $f : \mathbf{M} \longrightarrow \mathbb{R}^{n+1}$. Let $\tilde{\omega}$ be a fixed volume element on $\mathbb{R}^{n+1}$ such that $D\tilde{\omega} = 0$. Now define an induced volume element θ on $\mathbf{M}$ as

$$\theta(X_1, \cdots, X_n) = \tilde{\omega}(f_* X_1, \cdots, f_* X_n, \xi), \tag{2.6}$$

where ξ is the transversal vector field of the immersion $f : \mathbf{M} \longrightarrow \mathbb{R}^{n+1}$. In [2], Nomizu and Sasaki proved that the induced volume element θ satisfies, $\nabla_X \theta = \tau(X)\theta$ for each $X \in T_p(\mathbf{M})$. As a consequence, $\nabla\theta = 0$ if and only if $\tau = 0$, that is, $D_X \xi$ has only tangential part for vector fields X on $\mathbf{M}$.
An affine immersion $f : \mathbf{M} \longrightarrow \mathbb{R}^{n+1}$ is said to be equiaffine if $D_X \xi$ has only the tangential part, that is, the transversal connection form $\tau = 0$. An affine immersion $f : \mathbf{M} \longrightarrow \mathbb{R}^{n+1}$ of codimension one is called non-degenerate if the second fundamental form h is non-degenerate.

Let $f : \mathbf{M} \longrightarrow \mathbb{R}^{n+1}$ be an immersion of codimension one and ξ be an arbitrary transversal vector field on $\mathbf{M}$. Then,

$$T_{f(x)}(\mathbb{R}^{n+1}) = f_*(T_x \mathbf{M}) \bigoplus Span\{\xi_x\}, \tag{2.7}$$

for every $x \in \mathbf{M}$. Then, $D_X f_* Y$ is decomposed as

$$D_X f_* Y = f_*(\nabla_X Y) + h(X, Y)\xi, \tag{2.8}$$

for X, Y in $\mathcal{X}(\mathbf{M})$. Here $f_*(\nabla_X Y)$ is the notation for the tangential part of $D_X f_* Y$. Now, a connection ∇ is obtained on $\mathbf{M}$ defined by (2.8). This connection is called the induced connection on $\mathbf{M}$ with respect to ξ. Immersion (f, ξ) is said to be totally umbilical if S is proportional to the identity operator and $\tau = 0$. Note that (f, ξ) is totally umbilical if and only if the affine mean curvature $L = \frac{1}{n} trace(S)$ is constant and $\xi + Lx$ is constant along $\mathbf{M}$. Let $(\overline{f}, \overline{\xi})$ be another immersion of $\mathbf{M}$, $\overline{\nabla}$ and $\overline{h}$ be its induced connection and second fundamental form respectively. We say that $(\overline{f}, \overline{\xi})$ is dual to (f, ξ) if $h = \overline{h}$ on $\mathbf{M}$ and $\overline{\nabla}$ is dual to ∇ with respect to h.

For a non-degenerate totally umbilical immersion (f, ξ), the dual is constructed as follows.

The Legendre Transformation Let (f, ξ) be a non-degenerate totally umbilical affine immersion with $L = 0$. The local coordinates $(u^1, u^2, ...u^n)$ of $\mathbf{M}$ and the coordinates $(x^1, x^2, ...x^{n+1})$ of $\mathbb{R}^{n+1}$ be such that

$$\begin{aligned} x^i &= u^i \ (1 \leq i \leq n). \\ x^{n+1} &= \phi(u^1, u^2..., u^n), \text{where } \phi : \mathbf{M} \to \mathbb{R}. \\ \xi &= (0, 0....0, 1). \end{aligned}$$

Define the dual affine immersion $(\overline{f}, \overline{\xi})$ by

$$\begin{aligned} \overline{f}^i &= \frac{\partial \phi}{\partial u^i} \ (1 \leq i \leq n). \\ \overline{f}^{n+1} &= \sum_{i=1}^{n} u^i \overline{f}^i - \phi. \\ \overline{\xi} &= (0, 0....0, 1). \end{aligned}$$

Conormal Transformation Let (f, ξ) be a non-degenerate totally umbilical affine immersion with $L \neq 0$. Denote the dual space of $\mathbb{R}^{n+1}$ by $\mathbb{R}_{n+1}$. Define the affine immersion $(\overline{f}, \overline{\xi})$ of $\mathbf{M}$ into $\mathbb{R}_{n+1}$ as follows

$$\begin{aligned} \overline{f}(f_* X) &= 0 \ for \ X \in \ \mathcal{X}(\mathbf{M}). \\ \overline{f}(\xi) &= L. \\ \overline{\xi} &= -Lf. \end{aligned}$$

Note that in both the cases $(\overline{f}, \overline{\xi})$ is dual to (f, ξ).

Let $f : \mathbf{M} \longrightarrow \mathbb{R}^{n+1}$ be an affine immersion of codimension one and ξ be an arbitrary transversal vector field on $\mathbf{M}$. Let ∇ be the induced connection on $\mathbf{M}$ and R denote the curvature of $\mathbf{M}$ with respect to ∇. Then, the fundamental equations are [2]

$$\begin{aligned} R(X,Y)Z &= h(Y,Z)SX - h(X,Z)SY, && (2.9)\\ (\nabla_X h)(Y,Z) + \tau(X)h(Y,Z) &= (\nabla_Y h)(X,Z) + \tau(Y)h(X,Z), && (2.10)\\ (\nabla_Y S)(X) - \tau(Y)SX &= (\nabla_X S)(Y) - \tau(X)SY, && (2.11)\\ d\tau(X,Y) &= h(SY,X) - h(Y,SX). && (2.12) \end{aligned}$$

The equation (2.9) is called the Gauss equation, (2.10) and (2.11) are called the Codazzi equations for h and S, respectively. The equation (2.12) is called the Ricci equation.

Let $f : \mathbf{M} \longrightarrow \mathbb{R}^{n+1}$ be an affine immersion of codimension one and ξ be an arbitrary transversal vector field on $\mathbf{M}$. Let ∇ be the induced connection on $\mathbf{M}$ with respect to ξ. If f is non-degenerate and equiaffine, from the fundamental equations

$$(\nabla_X h)(Y,Z) = (\nabla_Y h)(X,Z) \tag{2.13}$$

(M, ∇, h) becomes a statistical manifold. We call (M, ∇, h) the statistical manifold realized in $\mathbb{R}^{n+1}$. In [6], Dillen et al. proved that a simply connected statistical manifold $(\mathbf{M}, \nabla, h)$ can be realized in $\mathbb{R}^{n+1}$ if and only if $\overline{\nabla}$ is projectively flat connection with symmetric Ricci tensor. As a consequence of this result we have

Proposition 2.1. *[4] A simply connected statistical manifold can be realized in $\mathbb{R}^{n+1}$ if and only if it is 1-conformally flat.*

Proof. A statistical manifold $(\mathbf{M}, \nabla, h)$ is 1-conformally flat if and only if the dual connection $\overline{\nabla}$ is projectively flat with symmetric Ricci tensor. Then, the proposition follows from the result mentioned above by Dillen et al. [6]. □

Now, we discuss two important theorems by Kurose [4] which describe the relation between affine immersions and statistical manifolds.

Theorem 2.1. *[4] Let $(\mathbf{M}, \nabla, \overline{\nabla}, g)$ be a simply connected and connected statistical manifold of dimension n. If $(\mathbf{M}, \nabla, \overline{\nabla}, g)$ has constant curvature, then there exist affine immersions (f, ξ) of $(\mathbf{M}, \nabla)$ and $(\overline{f}, \overline{\xi})$ of $(\mathbf{M}, \overline{\nabla})$ such that their second fundamental forms are equal to g. Moreover each of them is uniquely determined up to an affine transformation of $\mathbb{R}^{n+1}$ and $(\overline{f}, \overline{\xi})$ is obtained from (f, ξ) by the Legendre transformation or conormal transformation.*

Proof. Let $(\mathbf{M}, \nabla, \overline{\nabla}, g)$ be a simply connected and connected statistical manifold with constant curvature k. Define h, S and τ as follows:

$$\begin{aligned} h &= g. \\ S &= kI. \\ \tau &= 0. \end{aligned}$$

Then, $\nabla, h, S,$ and τ satisfy the equations (2.9), (2.10),(2.11) and (2.12). Then, there exists a non-degenerate eqiaffine immersion (f, ξ) of $(\mathbf{M}, \nabla)$ such that second fundamental form equal to g. The corresponding dual immersion $(\overline{f}, \overline{\xi})$ can be obtained using the Legendre transformation or Conormal transformation. □

Theorem 2.2. *[4] Let $(\mathbf{M}, \nabla, \overline{\nabla}, g)$ be a connected statistical manifold of dimension $n \geq 3$. If there exists an affine immersion of $(\mathbf{M}, \nabla)$ and an affine immersion of $(\mathbf{M}, \overline{\nabla})$ such that their second fundamental forms are equal to g, then $(\mathbf{M}, \nabla, \overline{\nabla}, g)$ has constant curvature.*

Proof. Let $(\mathbf{M}, \nabla, \overline{\nabla}, g)$ be a connected statistical manifold of dimension $n \geq 3$. Let $(\overline{f}, \overline{\xi})$ be the dual affine immersion of (f, ξ). Now, to show that $S = LI$. Since, ∇ and $\overline{\nabla}$ are dual with respect to g

$$g(R^{\nabla}(X, Y)Z, W) = -g(Z, R^{\overline{\nabla}}(X, Y)W), \tag{2.14}$$

for X, Y, Z and W in $\mathcal{X}(\mathbf{M})$. Then,

$$\begin{aligned} g(Y, Z)g(SX, W) - g(X, Z)g(SY, W) &= g(Y, W)g(\overline{S}X, Z) \\ &\quad -g(X, W)g(\overline{S}Y, Z). \end{aligned} \tag{2.15}$$

Set $L = \frac{1}{n}tr(S)$ and $\overline{L} = \frac{1}{n}tr(\overline{S})$. Now, taking trace in X and W components in (2.15) we get

$$nLg(Y, Z) - g(S(Y), Z) + g(\overline{S}Y, Z) - ng(\overline{S}Y, Z) = 0. \tag{2.16}$$

Again, taking trace in Y and Z components,

$$L = \overline{L}.$$

Also, from (2.16)

$$nLI = S + (n-1)\overline{S}. \tag{2.17}$$

Since equation (2.15) is symmetric in S and $\overline{S}$

$$nLI = \overline{S} + (n-1)S. \tag{2.18}$$

Equations (2.17) and (2.18) imply $S = LI$ for $n \geq 3$. Hence, $(\mathbf{M}, \nabla, \overline{\nabla}, g)$ has constant curvature. □

2.2 Immersions into Statistical Manifolds

In this section, we first discuss about the statistical submanifolds and the fundamental equations associated with it [7]. Uohashi et al. [8] proved that a 1-conformally flat statistical manifold of dimension $n \geq 2$ can be locally realized as a submanifold of a flat statistical manifold of dimension $(n+1)$. Then, we consider manifolds immersed in statistical manifolds and obtained a necessary and sufficient condition for the inherited statistical manifold structures to be dual to each other. Statistical immersion is defined and then proved a necessary condition for a statistical manifold to be a statistical hypersurface. Its converse is also proved [9]. Furuhata [10] has proved a necessary and sufficient condition for minimal immersion of a statistical manifold into $\mathbb{R}^{n+1}$. We proved a necessary and sufficient condition for a statistical immersion into a dually flat statistical manifold of codimension one to be minimal. Also, a necessary condition is obtained for minimal statistical immersion of statistical manifolds equipped with α-connections [11].

Let $(\mathbf{M}, g)$ be a submanifold of a statistical manifold $(\tilde{\mathbf{M}}, \tilde{\nabla}, \overline{\tilde{\nabla}}, \tilde{g})$ with the induced metric g, that is, the metric g is the restriction of $\tilde{g}$ onto the tangent space of $\mathbf{M}$. Let ∇ be the affine connection on $\mathbf{M}$ defined by

$$\nabla_X Y = \pi(\tilde{\nabla}_X Y), \tag{2.19}$$

where π is the orthogonal projection of $T\tilde{\mathbf{M}}$ onto $T\mathbf{M}$. That is, for each p in $\mathbf{M}$, $\pi_p : T_p(\tilde{\mathbf{M}}) \longrightarrow T_p(\mathbf{M})$ is a linear map such that $\pi_p(V) = V$, for all $V \in T_p(\mathbf{M})$. Then, $(\mathbf{M}, \nabla, g)$ becomes a statistical manifold called the induced statistical submanifold. Similary, define the connection $\overline{\nabla}$ induced by the dual connection $\overline{\tilde{\nabla}}$. In [7], Vos proved that (∇, g) and $(\overline{\nabla}, g)$ are dual statistical structures on $\mathbf{M}$.

Definition 2.4. Let $\mathbf{M}$ be a submanifold of $\tilde{\mathbf{M}}$. The statistical manifold $(\mathbf{M}, \nabla, \overline{\nabla}, g)$ is called the statistical submanifold of $(\tilde{\mathbf{M}}, \tilde{\nabla}, \overline{\tilde{\nabla}}, \tilde{g})$ if g, ∇ and $\overline{\nabla}$ coincide with the induced structures.

The corresponding Gauss formulae are [7]

$$\tilde{\nabla}_X Y = \nabla_X Y + \alpha(X, Y), \tag{2.20}$$
$$\overline{\tilde{\nabla}}_X Y = \overline{\nabla}_X Y + \overline{\alpha}(X, Y), \tag{2.21}$$

where α and $\overline{\alpha}$ are symmetric bilinear forms, called the affine fundamental forms (called the imbedding curvature tensors by Amari [31]) of $\mathbf{M}$ in $\tilde{\mathbf{M}}$ for $\tilde{\nabla}$ and for $\overline{\tilde{\nabla}}$, respectively.

Let ξ be a normal vector field on $\mathbf{M}$, define the shape operators S and $\overline{S}$ by

$$\begin{aligned} g(S_\xi X, Y) &= \tilde{g}(\alpha(X, Y), \xi), && (2.22)\\ g(\overline{S}_\xi X, Y) &= \tilde{g}(\overline{\alpha}(X, Y), \xi), && (2.23) \end{aligned}$$

where X and Y are in $\mathcal{X}(\mathbf{M})$. The corresponding Weingarten formulae are given as follows [7]

$$\begin{aligned} \tilde{\nabla}_X \xi &= -\overline{S}_\xi X + \nabla^{\perp}_X Y, && (2.24)\\ \overline{\tilde{\nabla}}_X \xi &= -S_\xi X + \overline{\nabla}^{\perp}_X Y, && (2.25) \end{aligned}$$

where $\nabla^{\perp}$ and $\overline{\nabla}^{\perp}$ are affine connections on normal bundle $T\mathbf{M}^{\perp}$. In addition, the fundamental equations relating the curvatures are [7]

$$\begin{aligned} \tilde{g}(\tilde{R}(X, Y)Z, W) &= g(R(X, Y)Z, W) + \tilde{g}(\alpha(X, Z), \overline{\alpha}(Y, W)) \\ &\quad -\tilde{g}(\overline{\alpha}(X, W), \alpha(Y, Z)), && (2.26)\\ (\tilde{R}(X, Y)Z)^{\perp} &= \nabla^{\perp}_X \alpha(Y, Z) - \alpha(\nabla_X Y, Z) - \alpha(Y, \nabla_X Z) \\ &\quad -\{\nabla^{\perp}_Y \alpha(X, Z) - \alpha(\nabla_Y X, Z) - \alpha(X, \nabla_Y Z)\}, && (2.27)\\ \tilde{g}(R^{\perp}(X, Y)\xi, \eta) &= \tilde{g}(\tilde{R}(X, Y)\xi, \eta) + g([\overline{S}_\xi, S_\eta](X), Y), && (2.28) \end{aligned}$$

where $R^{\perp}$ is the Riemannian curvature tensor on $T\mathbf{M}^{\perp}$, ξ, η are normal vector fields on $\mathbf{M}$ and $[\overline{S}_\xi, S_\eta] = \overline{S}_\xi S_\eta - S_\eta \overline{S}_\xi$. The equation (2.26) is called the *Gauss equation*, (2.27) is called the *Codazzi equation* and (2.28) is called the *equation of Ricci*.

In [8], Uohashi et al. introduced the statistical submanifold realized in a statistical manifold, which is nothing but a statistical submanifold of $(\tilde{\mathbf{M}}, \tilde{\nabla}, \tilde{g})$ defined above. They proved that, for a Hessian domain $(\Omega, \tilde{D}, \tilde{g} = \tilde{D}d\phi)$, n-dimensional level surfaces of ϕ are 1-conformally flat statistical submanifolds of $(\Omega, \tilde{D}, \tilde{g} = \tilde{D}d\phi)$. Also,

Theorem 2.3. *[8] An arbitrary* 1*-conformally flat statistical manifold of dimension* $n \geq$

2 *with a Riemannian metric can be locally realized as a submanifold of a flat statistical manifold of dimension* $(n+1)$.

Proof. Let $(\mathbf{M}, \nabla, g)$ be a 1-conformally flat statistical manifold of dimension $n \geq 2$ with a Riemannian metric g. Then, by proposition (2.1) $(\mathbf{M}, \nabla, g)$ can be realized in $\mathbb{R}^{n+1}$ by non-degenerate equiaffine immersion (f, ξ). Let U be a simply connected open subset of $\mathbf{M}$ and $\epsilon \geq 0$, define a function ϕ on $\tilde{U} = \cup_{q \in U}\{f(q) \oplus (-\epsilon, \epsilon).\xi_q\}$ by

$$\phi(p) = e^{-t} \text{ for } p = f(p_0) + t\xi_{p_0}, \; p_0 \in U, t \in (-\epsilon, \epsilon). \tag{2.29}$$

It is enough to show that (U, ∇, g) is a statistical submanifold of the flat statistical manifold $(\tilde{U}, \tilde{D}, \tilde{D}d\phi)$. For $X, Y \in \mathcal{X}(U)$, we have $d\phi(X) = 0$, $d\phi(\xi) = -1$. Then,

$$\begin{aligned}
(\tilde{D}_X d\phi)(Y) &= X(d\phi(Y)) - d\phi(\tilde{D}_X Y) \\
&= -d\phi(\nabla_X Y + g(X, Y)\xi) \\
&= -g(X, Y)d\phi(\xi) \\
&= g(X, Y).
\end{aligned}$$

Thus, (U, ∇, g) is a statistical submanifold of $(\tilde{U}, \tilde{D}, \tilde{D}d\phi)$.

Let E be a vector field on $\tilde{U}$ whose value is ξ_{p_0} at $p = f(p_0) + t\xi_{p_0}$. Then, on $f(U)$

$$E(d\phi(E)) = 1, \tilde{D}_E E = 0,$$

and

$$(\tilde{D}_E d\phi)(E) = 1.$$

Thus, $(\tilde{D}_E d\phi)_{f(p_0)}$ is positive definite for $p_0 \in U$. Since ϕ is continuous, $\tilde{D}d\phi$ is a Riemannian metric on $\tilde{U}$ for a small ϵ. Hence, $(\tilde{U}, \tilde{D}, \tilde{D}d\phi)$ is a flat statistical manifold. $\square$

Next, we discuss the geometry of an n-dimensional smooth manifold immersed into a statistical manifold of dimension $(n+1)$, such manifolds are known as statistical hypersurfaces.

Let $\mathbf{M}$ be an n-dimensional manifold and $(\tilde{\mathbf{M}}, \tilde{\nabla}, \tilde{g})$ be an $(n+1)$-dimensional statistical manifold. Let $f : \mathbf{M} \longrightarrow \tilde{\mathbf{M}}$ be an immersion. The induced metric $f^*\tilde{g}$ and the induced connection ∇ on $\mathbf{M}$ are defined as

$$f^*\tilde{g}(X, Y) = \tilde{g}(f_*X, f_*Y), \tag{2.30}$$

$$f^*\tilde{g}(\nabla_X Y, Z) \;=\; \tilde{g}(\tilde{\nabla}_X f_* Y, f_* Z), \tag{2.31}$$

for $X, Y, Z \in \mathcal{X}(\mathbf{M})$.

Note. The pullback $f^*\tilde{g}$ of $\tilde{g}$ is called the induced metric. Also, ∇ defined by (2.31) satisfies all the properties of an affine connection.

Note that from the above definition (2.30) and (2.31),

$$\begin{aligned}
(\nabla_X f^*\tilde{g})(Y, Z) &= X f^*\tilde{g}(Y, Z) - f^*\tilde{g}(\nabla_X Y, Z) - f^*\tilde{g}(Y, \nabla_X Z) \\
&= X\tilde{g}(f_*Y, f_*Z) - \tilde{g}(\tilde{\nabla}_X f_*Y, f_*Z) - \tilde{g}(f_*Y, \tilde{\nabla}_X f_*Z) \\
&= (\tilde{\nabla}_X \tilde{g})(f_*Y, f_*Z).
\end{aligned}$$

Thus, $(\mathbf{M}, \nabla, f^*\tilde{g})$ is also a statistical manifold.

Let $\mathbf{M}$ be an n-dimensional manifold and $(\tilde{\mathbf{M}}, \tilde{\nabla}, \overline{\tilde{\nabla}}, \tilde{g})$ be an $(n+1)$-dimensional statistical manifold. Let $f : \mathbf{M} \longrightarrow \tilde{\mathbf{M}}$ be an immersion into a statistical manifold of codimension one with the unit normal vector field ξ along f. Then, for each $p \in \mathbf{M}$

$$T_{f(p)}(\tilde{\mathbf{M}}) = f_*(T_p(\mathbf{M})) + span\{\xi_p\}. \tag{2.32}$$

Also, the equations connecting various tensors on $\mathbf{M}$ and $\tilde{\mathbf{M}}$ are

$$\begin{aligned}
\tilde{\nabla}_X f_* Y &= f_*(\nabla^{\top}_X Y) + h(X, Y)\xi, &(2.33)\\
\overline{\tilde{\nabla}}_X f_* Y &= f_*(\overline{\nabla}^{\top}_X Y) + \overline{h}(X, Y)\xi, &(2.34)\\
\tilde{\nabla}_X \xi &= -f_*(SX) + \tau(X)\xi, &(2.35)\\
\overline{\tilde{\nabla}}_X \xi &= -f_*(\overline{S}X) + \overline{\tau}(X)\xi, &(2.36)
\end{aligned}$$

for $X, Y \in \mathcal{X}(\mathbf{M})$, where $h(X, Y)$ and $\overline{h}(X, Y)$ are symmetric bilinear forms on tangent space $T_p(\mathbf{M})$ for p in $\mathbf{M}$, called the affine fundamental forms. The tensors S and $\overline{S}$ are tensor fields of type $(1, 1)$ and $\tau, \overline{\tau}$ are 1-forms. The tensor S ($\overline{S}$) is called the shape operator and τ ($\overline{\tau}$) is the transversal connection form for f and the induced connections $\nabla^{\top}$ ($\overline{\nabla}^{\top}$). The equations (2.33), (2.34) are called the Gauss formulae and (2.35), (2.36) are called the Weingarten formulae [43], [7].

The connection $\nabla^{\top}$ induced by the Gauss formula coincides with the induced connection ∇ defined by (2.31) on $\mathbf{M}$. So, we write ∇ instead of $\nabla^{\top}$. Thus, on $\mathbf{M}$ we have induced connections ∇ and $\overline{\nabla}$ and the second fundamental forms h and $\overline{h}$. Using the Gauss and the Weingarten formulae we have the following observations.

(a). The induced connections ∇ and $\overline{\nabla}$ on $\mathbf{M}$ are dual with respect to the induced metric $f^*\tilde{g}$.
(b). For vector fields $X, Y \in \mathcal{X}(\mathbf{M})$

(i) $h(X,Y) = f^*\tilde{g}(\overline{S}X, Y)$.

(ii) $\overline{h}(X,Y) = f^*\tilde{g}(SX, Y)$.

(iii) $\tau(X) + \overline{\tau}(X) = 0$.

Now, suppose $(\tilde{\mathbf{M}}, \tilde{\nabla}, \overline{\tilde{\nabla}}, \tilde{g})$ has constant curvature $\tilde{k}$, then the fundamental equations are [43], [7]

$$\begin{aligned} R^{\nabla}(X,Y)Z &= \tilde{k}\{f^*\tilde{g}(Y,Z)X - f^*\tilde{g}(X,Z)Y\} + h(Y,Z)SX \\ &\quad -h(X,Z)SY, \qquad (2.37) \\ (\nabla_X h)(Y,Z) + \tau(X)h(Y,Z) &= (\nabla_Y h)(X,Z) + \tau(Y)h(X,Z), \qquad (2.38) \\ (\nabla_X S)(Y) - \tau(X)SY &= (\nabla_Y S)(X) - \tau(Y)SX, \qquad (2.39) \\ h(X,SY) - h(SX,Y) &= d\tau(X,Y), \qquad (2.40) \end{aligned}$$

where R^{∇} denotes the curvature tensor with respect to ∇ in $\mathbf{M}$. Equation (2.37) is called the Gauss equation, (2.38) and (2.39) are called the Codazzi equation for h and the Codazzi equation for S, respectively. The equation (2.40) is called the Ricci equation. Similarly, one can write the fundamental equations with respect to the dual connection also.

Definition 2.5. Let $f : \mathbf{M} \longrightarrow \tilde{\mathbf{M}}$ be an immersion of codimension one. Then, f is said to be non-degenerate if the second fundamental form h is non-degenerate and f is equiaffine if the transversal connection form τ is zero.

Remark 2.1. For a non-degenerate equiaffine immersion $f : \mathbf{M} \longrightarrow \tilde{\mathbf{M}}$ with $\tilde{\mathbf{M}}$ has constant curvature, from equation (2.38) we get $(\mathbf{M}, \nabla, h)$ is a statistical manifold. Similarly, $(\mathbf{M}, \overline{\nabla}, \overline{h})$ is also a statistical manifold, where $\overline{\nabla}$ and $\overline{h}$ are the connection and the second fundamental form obtained using the dual connection $\overline{\tilde{\nabla}}$ of $(\tilde{\mathbf{M}}, \tilde{\nabla}, \overline{\tilde{\nabla}}, \tilde{g})$. Note that, in this case an equation similar to equation (2.38) called the Codazzi equation for $\overline{h}$ can be written.

Definition 2.6. Statistical manifolds $(\mathbf{M}, \nabla, h)$ and $(\mathbf{M}, \overline{\nabla}, \overline{h})$ are said to be dual to each other if $h = \overline{h}$ and the connections ∇, $\overline{\nabla}$ are dual with respect to h.

Now, we prove a necessary and sufficient condition for the inherited statistical manifolds $(\mathbf{M}, \nabla, h)$ and $(\mathbf{M}, \overline{\nabla}, \overline{h})$ to be dual to each other.

Theorem 2.4. *Let* M *be an* n*-dimensional manifold and* $(\tilde{\mathrm{M}}, \tilde{\nabla}, \tilde{g})$ *be an* $(n+1)$*-dimensional statistical manifold with constant curvature* $\tilde{k}$*. Let* $f : \mathrm{M} \longrightarrow \tilde{\mathrm{M}}$ *be a non-degenerate, equiaffine immersion of codimension one. Then,* (M, ∇, h) *and* $(\mathrm{M}, \overline{\nabla}, \overline{h})$ *are dual to each other if and only if* $S = \overline{S} = \lambda I$ *for some constant* λ*. Moreover* $h = \lambda f^* \tilde{g}$*.*

Proof. Suppose $S = \overline{S} = \lambda I$ for some constant λ. From the above observation (b)

$$\begin{aligned} h(X, Y) &= f^* \tilde{g}(\overline{S} X, Y) \\ &= \lambda f^* \tilde{g}(X, Y). \end{aligned}$$

Similarly, $\overline{h}(X, Y) = \lambda f^* \tilde{g}(X, Y)$, thus $h = \overline{h} = \lambda f^* \tilde{g}$. Since ∇ and $\overline{\nabla}$ are dual with respect to $f^* \tilde{g}$, the statistical manifolds (M, ∇, h) and $(\mathrm{M}, \overline{\nabla}, \overline{h})$ are dual to each other.

Conversely, let (M, ∇, h) and $(\mathrm{M}, \overline{\nabla}, \overline{h})$ be dual to each other. Then $h = \overline{h}$ and ∇, $\overline{\nabla}$ are dual with respect to h. So

$$Zh(X, Y) = h(\nabla_Z X, Y) + h(X, \overline{\nabla}_Z Y). \tag{2.41}$$

Now, consider

$$\begin{aligned} Z f^* \tilde{g}(SX, Y) &= f^* \tilde{g}(\nabla_Z SX, Y) + f^* \tilde{g}(SX, \overline{\nabla}_Z Y) \\ &= f^* \tilde{g}(\nabla_Z SX, Y) + h(X, \overline{\nabla}_Z Y). \end{aligned} \tag{2.42}$$

Since $h(X, Y) = f^* \tilde{g}(\overline{S} X, Y) = f^* \tilde{g}(SX, Y) = \overline{h}(X, Y)$, from (2.41) and (2.42) we get

$$\nabla_Z SX = S(\nabla_Z X),$$

which implies $(\nabla_Z S) X = 0$, then $S = \lambda I$ for some constant λ.
Therefore $S = \overline{S} = \lambda I$. Also, note that the induced metric $h = \lambda f^* \tilde{g}$. □

Remark 2.2. Let $(\tilde{\mathrm{M}}, \tilde{\nabla}, \tilde{g})$ be an $(n+1)$-dimensional statistical manifold. If $f : \mathrm{M} \longrightarrow \tilde{\mathrm{M}}$ is an immersion of codimension one, then $(\mathrm{M}, \nabla, \overline{\nabla}, f^* \tilde{g})$ is a statistical manifold. For a non-degenerate equiaffine immersion $f : \mathrm{M} \longrightarrow \tilde{\mathrm{M}}$ of codimension one with $\tilde{\mathrm{M}}$ having constant curvature, $(\mathrm{M}, \nabla, \overline{\nabla}, h)$ is a statistical manifold if and only if $S = \overline{S} = \lambda I$ for some constant λ.

Definition 2.7. Let $(\tilde{\mathrm{M}}, \tilde{\nabla}, \tilde{g})$ and (M, ∇, g) be statistical manifolds. An immersion $f : \mathrm{M} \longrightarrow \tilde{\mathrm{M}}$ is called a statistical immersion if the metric g on M coincides with the induced metric and the connection ∇ on M coincides with the induced connection.

Note. Fundamental equations can be written for statistical immersions of codimension one. In this case, the Gauss equation (2.37) reduces to

$$\begin{aligned} R^{\nabla}(X,Y)Z &= \tilde{k}\{g(Y,Z)X - g(X,Z)Y\} + h(Y,Z)SX \\ &\quad -h(X,Z)SY, \end{aligned} \tag{2.43}$$

and all other equations (2.38), (2.39) and (2.40) remain the same.

Example 2.3. Let $H = \{(x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} > 0\}$ be the upper half space of constant curvature (-1) with Riemannian metric $\tilde{g} = (x_{n+1})^{-2}\sum_{i=1}^{n+1}(dx_i)^2$. Define an affine connection $\tilde{\nabla}$ on H by the following relations

$$\tilde{\nabla}_{\frac{\partial}{\partial x_{n+1}}}\frac{\partial}{\partial x_{n+1}} = (x_{n+1})^{-1}\frac{\partial}{\partial x_{n+1}}, \quad \tilde{\nabla}_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j} = 2\delta_{ij}(x_{n+1})^{-1}\frac{\partial}{\partial x_{n+1}}, \tag{2.44}$$

$$\tilde{\nabla}_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_{n+1}} = \tilde{\nabla}_{\frac{\partial}{\partial x_{n+1}}}\frac{\partial}{\partial x_j} = 0, \tag{2.45}$$

where $i, j = 1, \cdots, n$. Then, $(H, \tilde{\nabla}, \tilde{g})$ is a Hessian manifold. For a constant $y_0 > 0$, define the map $f_0 : \mathbb{R}^n \longrightarrow H$ by $f_0\left((x_1, \cdots x_n)\right) = (x_1, \cdots, x_n, y_0)$. Then, f_0 is an immersion, (∇, g) be the statistical structure on $\mathbb{R}^n$ induced by f_0 from $(\tilde{\nabla}, \tilde{g})$. Then, $f_0 : (\mathbb{R}^n, \nabla, g) \longrightarrow (H, \tilde{\nabla}, \tilde{g})$ is a statistical immersion. Note that the transversal vector field $\xi = y_0\frac{\partial}{\partial x_{n+1}}$, the second fundamental forms $h = 2g$ and $\overline{h} = 0$. Also, the affine shape operators $S = 2I$, $\overline{S} = 0$ and the transversal connection forms $\tau = \overline{\tau} = 0$.

Definition 2.8. An n-dimensional statistical manifold $(\mathbf{M}, \nabla, g)$ is said to be realized in an $(n+1)$-dimensional statistical manifold $(\tilde{\mathbf{M}}, \tilde{\nabla}, \tilde{g})$ if there exists a non-degenerate equiaffine immersion $f : \mathbf{M} \longrightarrow \tilde{\mathbf{M}}$ such that the affine fundamental form h equal to g and the induced connection coincide with ∇.

Now, we prove

Theorem 2.5. *Let* $(\mathbf{M}, \nabla, \overline{\nabla}, g)$ *and* $(\tilde{\mathbf{M}}, \tilde{\nabla}, \overline{\tilde{\nabla}}, \tilde{g})$ *be simply connected and connected statistical manifolds of dimensions* n *and* $(n+1)$ *with constant curvatures* k *and* $\tilde{k}$, *respectively, then* $\mathbf{M}$ *is realized in* $\tilde{\mathbf{M}}$.

Proof. Define

$$h = g, \tag{2.46}$$

$$S = (k - \tilde{k})I, \tag{2.47}$$

$$\tau = 0. \tag{2.48}$$

Note that ∇, h, S, τ satisfy the fundamental equations (2.43), (2.38), (2.39) and (2.40). Then, there exists a non-degenerate equiaffine statistical immersion $(f, \xi) : \mathbf{M} \longrightarrow \tilde{\mathbf{M}}$ such that h, S, τ are the affine fundamental form, the affine shape operator and the transversal connection of (f, ξ), respectively. Hence, $\mathbf{M}$ is realized in $\tilde{\mathbf{M}}$. □

Also, we have

Theorem 2.6. *Let $(\tilde{\mathbf{M}}, \tilde{\nabla}, \overline{\tilde{\nabla}}, \tilde{g})$ be an $(n+1)$-dimensional connected statistical manifold with constant curvature $\tilde{k}$ and $(\mathbf{M}, \nabla, \overline{\nabla}, g)$ be a connected statistical manifold of dimension $n \geq 3$. If there exists a statistical immersion $(f, \xi) : (\mathbf{M}, \nabla, \overline{\nabla}, g) \longrightarrow (\tilde{\mathbf{M}}, \tilde{\nabla}, \overline{\tilde{\nabla}}, \tilde{g})$ such that $h = \overline{h} = g$, then $(\mathbf{M}, \nabla, \overline{\nabla}, g)$ has constant curvature.*

Proof. Since, by the Gauss equation

$$\begin{aligned} R^{\nabla}(X, Y)Z &= \tilde{k}\{g(Y, Z)X - g(X, Z)Y\} + h(Y, Z)SX \\ &\quad -h(X, Z)SY, \end{aligned}$$

it is enough to show that $S = \lambda I$ for some constant λ. Now, consider

$$\begin{aligned} R^{\overline{\nabla}}(X, Y)W &= \tilde{k}\{g(Y, W)X - g(X, W)Y\} + \overline{h}(Y, W)\overline{S}X \\ &\quad -\overline{h}(X, W)\overline{S}Y. \end{aligned}$$

∇ and $\overline{\nabla}$ are dual with respect to g, so

$$g(R^{\nabla}(X, Y)Z, W) = -g(Z, R^{\overline{\nabla}}(X, Y)W), \tag{2.49}$$

for X, Y, Z and W in $\mathcal{X}(\mathbf{M})$. Then,

$$\begin{aligned} g(Y, Z)g(SX, W) - g(X, Z)g(SY, W) &= g(Y, W)g(\overline{S}X, Z) \\ &\quad -g(X, W)g(\overline{S}Y, Z). \end{aligned} \tag{2.50}$$

Set $L = \frac{1}{n}tr(S)$ and $\overline{L} = \frac{1}{n}tr(\overline{S})$. Now, taking trace in X and W components in (2.50) we get

$$nLg(Y, Z) - g(SY, Z) + g(\overline{S}Y, Z) - ng(\overline{S}Y, Z) = 0. \tag{2.51}$$

Again, taking trace in Y and Z components,

$$L = \overline{L}. \tag{2.52}$$

Also, from (2.51)

$$nLI = S + (n-1)\overline{S}. \tag{2.53}$$

Since equation (2.50) is symmetric in S and $\overline{S}$

$$nLI = \overline{S} + (n-1)S. \tag{2.54}$$

Equations (2.53) and (2.54) imply that $S = LI$ for $n \geq 3$. Hence, $(\mathbf{M}, \nabla, \overline{\nabla}, g)$ has constant curvature. $\square$

Theorem 2.7. *Let $f : (\mathbf{M}, \nabla, g) \longrightarrow (\tilde{\mathbf{M}}, \tilde{\nabla}, \tilde{g})$ be a non-degenerate equiaffine statistical immersion of codimension one. If $(\mathbf{M}, \nabla, h)$ and $(\mathbf{M}, \overline{\nabla}, \overline{h})$ are dual to each other, then $f : (\mathbf{M}, \nabla^{(\alpha)}, g) \longrightarrow (\tilde{\mathbf{M}}, \tilde{\nabla}^{(\alpha)}, \tilde{g})$ is a statistical immersion of codimension one with the Gauss equation $\tilde{\nabla}^{(\alpha)}_X f_* Y = f_*(\nabla^{(\alpha)}_X Y) + h(X, Y)\xi$, for $\alpha \in \mathbb{R}$.*

Proof. Consider

$$\begin{aligned}
f_*(\nabla^{(\alpha)}_X Y) &= f_*\left(\frac{1+\alpha}{2}(\nabla_X Y) + \frac{1-\alpha}{2}(\overline{\nabla}_X Y)\right) \\
&= \left(\frac{1+\alpha}{2}\right) f_*(\nabla_X Y) + \left(\frac{1-\alpha}{2}\right) f_*(\overline{\nabla}_X Y) \\
&= \left(\frac{1+\alpha}{2}\right)(\tilde{\nabla}_X f_* Y - h(X, Y)\xi) + \left(\frac{1-\alpha}{2}\right)(\overline{\tilde{\nabla}}_X f_* Y - \overline{h}(X, Y)\xi) \\
&= \left(\frac{1+\alpha}{2}\right)\tilde{\nabla}_X f_* Y + \left(\frac{1-\alpha}{2}\right)\overline{\tilde{\nabla}}_X f_* Y - \left(\frac{1+\alpha}{2}\right) h(X, Y)\xi \\
&\quad - \left(\frac{1-\alpha}{2}\right)\overline{h}(X, Y)\xi,
\end{aligned}$$

since $(\mathbf{M}, \nabla, h)$ and $(\mathbf{M}, \overline{\nabla}, \overline{h})$ are dual to each other

$$f_*(\nabla^{(\alpha)}_X Y) = \tilde{\nabla}^{(\alpha)}_X f_* Y - h(X, Y)\xi.$$

Then

$$\tilde{\nabla}^{(\alpha)}_X f_* Y = f_*(\nabla^{(\alpha)}_X Y) + h(X, Y)\xi, \text{for } \alpha \in \mathbb{R},$$

hence we get

$$\tilde{g}(\tilde{\nabla}^{(\alpha)}_X f_* Y, f_* Z) = g(\nabla^{(\alpha)}_X Y, Z)$$

So, $f : (\mathbf{M}, \nabla^{(\alpha)}, g) \longrightarrow (\tilde{\mathbf{M}}, \tilde{\nabla}^{(\alpha)}, \tilde{g})$ is a statistical immersion of codimension one with the Gauss equation

$$\tilde{\nabla}^{(\alpha)}_X f_*Y = f_*(\nabla^{(\alpha)}_X Y) + h(X,Y)\xi, \quad \text{for } \alpha \in \mathbb{R}.$$

$\square$

The notion of affine equivalence has been introduced by Dillen [44]. Two affine immersions $f_1 : (\mathbf{M}_1, \nabla^1) \longrightarrow (\mathbb{R}^{n+1}, D)$ and $f_2 : (\mathbf{M}_2, \nabla^2) \longrightarrow (\mathbb{R}^{n+1}, D)$ are said to be affine equivalent with respect a diffeomorphism $F : (\mathbf{M}_1, \nabla^1) \longrightarrow (\mathbf{M}_2, \nabla^2)$ if F preserves the connection and such that $f_2 \circ F = f_1$. He also proved that f_1 and f_2 are equivalent with respect to F if the following conditions are satisfied.

1. There exists a diffeomorphism $F : (\mathbf{M}_1, \nabla^1) \longrightarrow (\mathbf{M}_2, \nabla^2)$ that preserves the connection.

2. $h_2(F_*X, F_*Y) = h_1(X,Y)$.

3. $\tau^1(X) = \tau^2(F_*X)$.

4. $rank(h_1) = rank(h_2) > 1$.

Now, we have the following theorem for statistical manifolds.

Theorem 2.8. *Let $(\mathbf{M}_1, \nabla^1, h_1)$ and $(\mathbf{M}_2, \nabla^2, h_2)$ be two statistical manifolds realized in $\mathbb{R}^{n+1}$ with respect to two non-degenerate equiaffine immersions $f_1 : \mathbf{M}_1 \longrightarrow \mathbb{R}^{n+1}$ and $f_2 : \mathbf{M}_2 \longrightarrow \mathbb{R}^{n+1}$ with common transversal vector field ξ. Then, f_1 and f_2 are equivalent with respect to a diffeomorphism $F : (\mathbf{M}_1, \nabla^1) \longrightarrow (\mathbf{M}_2, \nabla^2)$ if and only if F is a statistical immersion.*

Proof. Suppose f_1 and f_2 are equivalent with respect to F, then by definition F is a diffeomrphism such that $f_2 \circ F = f_1$ and $\nabla^2_{F_*Z} F_*X = F_*(\nabla^1_Z X)$, for $X, Z \in \mathcal{X}(\mathbf{M_1})$. Since, $f_1 = f_2 \circ F$

$$\begin{aligned} f_{1*}X &= (f_2 \circ F)_*(X) \\ &= f_{2*}(F_*(X)), \end{aligned}$$

for $X \in \mathcal{X}(\mathbf{M_1})$. Now, consider

$$\begin{aligned} D_X(f_{2*}(F_*(Y))) &= D_{F_*X}(f_{2*}(F_*(Y))), \\ &= f_{2*}(\nabla^2_X F_*Y) + h_2(F_*X, F_*Y)\xi. \end{aligned} \tag{2.55}$$

Also,

$$\begin{aligned} D_X f_{1*} Y &= f_{1*}(\nabla^1_X Y) + h_1(X,Y)\xi, \\ &= f_{2*}(F_*(\nabla^1_X Y)) + h_1(X,Y)\xi. \end{aligned} \tag{2.56}$$

By equating the transversal part of (2.55) and (2.56) we get

$$h_2(F_*X, F_*Y) = h_1(X,Y).$$

Hence,

$$h_2(\nabla^2_{F_*X} F_*Y, F_*Z) = h_1(\nabla^1_X Y, Z).$$

So $F : \mathbf{M}_1 \longrightarrow \mathbf{M}_2$ is a statistical immersion.
Conversely, suppose there exist a statistical immersion $F : \mathbf{M}_1 \longrightarrow \mathbf{M}_2$ then

$$h_2(F_*X, F_*Y) = h_1(X,Y). \tag{2.57}$$

$$h_2(\nabla^2_{F_*X} F_*Y, F_*Z) = h_1(\nabla^1_X Y, Z). \tag{2.58}$$

Also, by the definition of isometric immersion F is a diffeomorphism. Now, it is enough to show that $F_*(\nabla^1_X Y) = \nabla^2_{F_*X} F_*Y$.
From (2.57) and (2.58) we have

$$\begin{aligned} h_2(\nabla^2_{F_*X} F_*Y, F_*Z) &= h_1(\nabla^1_X Y, Z) \\ &= h_2(F_*(\nabla^1_X Y), F_*Z), \end{aligned}$$

for $X, Y, Z \in \mathcal{X}(\mathbf{M})$. This implies $F_*(\nabla^1_X Y) = \nabla^2_{F_*X} F_*Y$.
Hence, f_1 and f_2 are equivalent with respect to F. □

2.2.1 Minimal Immersions

Statistical structure induced by the minimal affine immersions are studied by Furuhatha [10]. He obtained a necessary and sufficient condition for a statistical structure to be realized as a minimal affine hypersurface. In this subsection, we give a necessary and sufficient condition for the existence of a minimal statistical immersion. Also, obtained conditions for a statistical immersion to be minimal for statistical manifolds with α- connections [11].

Definition 2.9. An affine immersion $f : \mathrm{M} \longrightarrow \mathbb{R}^{n+1}$ is said to be a minimal affine immersion if the affine mean curvature $L = \frac{1}{n}tr(S)$ vanishes identically, where S is the affine shape operator.

Furhatha [10] proved that,

Theorem 2.9. *Let M be a simply connected, oriented manifold of dimension n and (∇, h) be a statistical structure on M. A necessary and sufficient condition for (∇, h) to be induced by a minimal affine immersion of M into $\mathbb{R}^{n+1}$ is the following:*

1. $\nabla Vol_h = 0$
2. (∇, h) *is dual-projectively flat*
3. *The scalar curvature $Scal^{(\nabla,h)}$ of (∇, h) vanishes identically, where $Scal^{(\nabla,h)} = 2L$.*

Definition 2.10. Let $f : \mathrm{M} \longrightarrow \tilde{\mathrm{M}}$ be a statistical immersion of codimension one. Then f is called a minimal immersion if the mean curvature $L = \frac{1}{n}trS$ vanishes identically.

Then, we have

Theorem 2.10. *Let $f : (\mathrm{M}, \nabla, g) \longrightarrow (\tilde{\mathrm{M}}, \tilde{\nabla}, \tilde{g})$ be a statistical immersion of codimension one and $(\tilde{\mathrm{M}}, \tilde{\nabla}, \overline{\tilde{\nabla}}, \tilde{g})$ be dually flat. Then, f is a minimal immersion if and only if the scalar curvature $Scal^{(\nabla,h)}$ vanishes identically.*

Proof. Since $(\tilde{\mathrm{M}}, \tilde{\nabla}, \overline{\tilde{\nabla}}, \tilde{g})$ is dually flat, we have $\tilde{k} = 0$. Then, the equation

$$R^{\nabla}(X,Y)Z = \tilde{k}\{g(Y,Z)X - g(X,Z)Y\} + h(Y,Z)SX - h(X,Z)SY$$

reduces to

$$R^{\nabla}(X,Y)Z = h(Y,Z)SX - h(X,Z)SY.$$

Then,

$$Ric^{\nabla}(Y,Z) = h(Y,Z)trS - h(SY,Z). \tag{2.59}$$

From the definition of the scalar curvature

$$Scal^{(\nabla,h)} = 2L,$$

where $L = \frac{1}{n}trS$. Then, f is a minimal immersion if and only if the scalar curvature $Scal^{(\nabla,h)}$ vanishes identically. □

Definition 2.11. Let $(\mathbf{M}, \nabla, \overline{\nabla}, g)$ be an n-dimensional statistical manifold. Then the difference tensor $K(X, Y)$ is defined by

$$K(X,Y) = \overline{\nabla}_X Y - \nabla_Y X, \quad for \ \ X, Y \in \mathcal{X}(\mathbf{M}). \tag{2.60}$$

Remark 2.3. From
$$\overset{g}{\nabla} = \frac{1}{2}(\nabla + \overline{\nabla}),$$

$$K(X,Y) = 2(\overset{g}{\nabla}_X Y - \nabla_Y X) = 2(\overline{\nabla}_X Y - \overset{g}{\nabla}_Y X), \text{ for } X, Y \in \mathcal{X}(\mathbf{M}). \text{ Also,}$$

$$\nabla^{\alpha}_X Y = \overset{g}{\nabla}_X Y - \frac{\alpha}{2} K(X,Y),$$

with

$$\nabla_X Y = \overset{g}{\nabla}_X Y - \tfrac{1}{2} K(X,Y), \quad \overline{\nabla}_X Y = \overset{g}{\nabla}_X Y + \tfrac{1}{2} K(X,Y).$$

Here $\overset{g}{\nabla}$ is the Levi-Civitta connection with respect to g.

Note. The curvature tensor $R^{\nabla^{\alpha}}$ for the α-connection ∇^{α} satisfies

$$\begin{aligned} R^{\nabla^{(\alpha)}}(X,Y)Z &= \frac{1+\alpha}{2} R^{\nabla}(X,Y)Z + \frac{1-\alpha}{2} R^{\overline{\nabla}}(X,Y)Z \\ &\quad + \frac{1-\alpha^2}{4}(K(Y, K(X,Z)) - K(X, K(Y,Z))). \end{aligned}$$

In local coordinates

$$\begin{aligned} \nabla_{\partial_i}\partial_j &= \sum_{\ell} \Gamma^{\ell}_{ij}\partial_{\ell} \\ R^{\nabla}(\partial_i, \partial_j)\partial_{\ell} &= \sum_k R^k_{\ell ij}\partial_k \end{aligned}$$

with
$$R^k_{\ell ij} = \partial_i \Gamma^k_{ij} - \partial_j \Gamma^k_{\ell i} + \sum_m (\Gamma^k_{mi}\Gamma^m_{ij} - \Gamma^m_{\ell i}\Gamma^k_{mj})$$

and

$$K(\partial_i, \partial_j) = \textstyle\sum_k K^k_{ij}\partial_k = \sum_k (\overline{\Gamma}^k_{ij} - \Gamma^k_{ij})\partial_k$$

with

$$
\begin{aligned}
K(\partial_i, K(\partial_j, \partial_\ell)) &= K(\partial_i, \sum_m K^m_{j\ell}\partial_m) \\
&= \sum_m K^m_{j\ell} K(\partial_i, \partial_m) \\
&= \sum_{m,k} K^m_{j\ell} K^k_{im} \partial_k.
\end{aligned}
$$

Also

$$
R^{(\alpha)k}_{\ell ij} = \frac{1+\alpha}{2} R^k_{\ell ij} + \frac{1-\alpha}{2} \overline{R}^k_{\ell ij} + \frac{1-\alpha^2}{4} (\sum_m K^m_{i\ell} K^k_{jm} - \sum_m K^k_{im} K^m_{j\ell}).
$$

The Ricci tensor $R^\alpha_{ij} = \sum_k R^{(\alpha)k}_{ikj}$ gives

$$
R^{(\alpha)}_{ij} = \frac{1+\alpha}{2} R_{ij} + \frac{1-\alpha}{2} \overline{R}_{ij} + \frac{1-\alpha^2}{4} (\sum_{m,k} K^m_{ik} K^k_{jm} - \sum_{m,k} K^m_{ij} K^k_{km})
$$

and the scalar curvature $Scal^{(\nabla,h)} = \sum_{j\ell} h^{j\ell} R^\nabla_{\ell,j}$.

Proposition 2.2. *[45] The scalar curvature $Scal^{(\nabla^\alpha,h)}$ for ∇^α is related to $Scal^{(\nabla,h)}$ as*

$$
Scal^{(\nabla^\alpha,h)} = Scal^{(\nabla,h)} + \frac{1-\alpha^2}{4} \mathcal{K}, \tag{2.61}
$$

where

$$
\mathcal{K} = \sum_{m,k,i,j} h^{ij} (K^m_{ik} K^k_{jm} - K^m_{ij} K^k_{km}). \tag{2.62}
$$

Proof. Consider $h(R^\nabla(X,Y)Z, W) = -h(Z, R^{\overline{\nabla}}(X,Y)W)$. Writing this equation in the component form gives

$$
\sum_k h_{km} R^k_{\ell ij} + \sum_k h_{\ell k} \overline{R}^k_{mij} = 0.
$$

Multiplying by h^{tm}, $h^{s\ell}$ and sum over ℓ, m indices,

$$
\sum_\ell h_{\ell s} R^t_{\ell ij} + \sum_m h_{mt} \overline{R}^s_{mij} = 0. \tag{2.63}
$$

Since ∇ is torsion free

$$
R^\nabla(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{\nabla_X Y} Z + \nabla_{\nabla_Y X} Z,
$$

in local coordinates

$$R^t_{\ell ij} = -R^t_{\ell ji}. \tag{2.64}$$

From (2.63) and (2.64)

$$Scal^{(\nabla,h)} = Scal^{(\overline{\nabla},h)}.$$

Now, consider

$$R^{(\alpha)}_{ij} = \frac{1+\alpha}{2}R_{ij} + \frac{1-\alpha}{2}\overline{R}_{ij} + \frac{1-\alpha^2}{4}(\sum_{m,k} K^m_{ik}K^k_{jm} - \sum_{m,k} K^m_{ij}K^k_{km}).$$

Multiplying by h^{ij} and sum over i, j,

$$Scal^{(\nabla^{(\alpha)},h)} = \frac{1+\alpha}{2}Scal^{(\nabla,h)} + \frac{1-\alpha}{2}Scal^{(\overline{\nabla},h)} + \frac{1-\alpha^2}{4}\mathcal{K},$$

where

$$\mathcal{K} = \sum_{m,k,i,j} h^{ij}(K^m_{ik}K^k_{jm} - K^m_{ij}K^k_{km}).$$

Since,

$$Scal^{(\nabla,h)} = Scal^{(\overline{\nabla},h)}$$

we have

$$Scal^{(\nabla^{\alpha},h)} = Scal^{(\nabla,h)} + \frac{1-\alpha^2}{4}\mathcal{K}.$$

□

Now, we prove a necessary condition for the minimal statistical immersion of statistical manifolds equipped with α-connections.

Theorem 2.11. *Let $f : (\mathbf{M}, \nabla^{(\alpha)}, g) \longrightarrow (\tilde{\mathbf{M}}, \tilde{\nabla}^{(\alpha)}, \tilde{g})$ be a non-degenerate statistical immersion of codimension one. Then, f is a minimal immersion if $f : (\mathbf{M}, \nabla, g) \longrightarrow (\tilde{\mathbf{M}}, \tilde{\nabla}, \tilde{g})$ is a minimal immersion and $\nabla_X Y = \overline{\nabla}_X Y$.*

Proof. Since $f : (\mathbf{M}, \nabla, g) \longrightarrow (\tilde{\mathbf{M}}, \tilde{\nabla}, \tilde{g})$ is a minimal immersion, we have

$$Scal^{(\nabla,h)} = 0.$$

Then, from the above proposition

$$Scal^{(\nabla^{\alpha},h)} = \frac{1-\alpha^2}{2}\mathcal{K}.$$

Since $\nabla_X Y = \overline{\nabla}_X Y$, we get $Scal^{(\nabla^{\alpha},h)} = 0$. Then, from the theorem (2.10) $f : (\mathbf{M}, \nabla^{(\alpha)}, g) \longrightarrow (\tilde{\mathbf{M}}, \tilde{\nabla}^{(\alpha)}, \tilde{g})$ is a minimal immersion for $\alpha \in \mathbb{R}$. $\square$

2.3 Centro-affine Immersion of Codimension Two

In this section, centro-affine immersion into $\mathbb{R}^{n+2}$ and the fundamental equations of it are discussed first [12], [13]. Then, dependence on the change of transversal vector field and change in an immersion are discussed [13]. Also, we give a detailed proof of 1-conformal equivalence and (-1)-conformal equivalence of statistical manifold structures in the case of centro-affine immersions into $\mathbb{R}^{n+2}$. We define centro-affine immersions into a dually flat statistical manifold of codimension two and give a necessary and sufficient condition for the inherited statistical manifold structures to be dual to each other. Then, show that the inherited statistical manifold structure is conformally-projectively flat in the case of non-degenerate, centro-affine, equiaffine immersion into a dually flat statistical manifold of codimesion two [9].

Let $\mathbf{M}$ be an n-dimensional manifold and $f : \mathbf{M} \longrightarrow \mathbb{R}^{n+2}$ be an immersion. Let D be the standard flat affine connection on $\mathbb{R}^{n+2}$ and $\eta = \sum_{i=1}^{n+2} x^i \frac{\partial}{\partial x^i}$ be the radial vector field of $\mathbb{R}^{n+2} - \{0\}$, where $\{x^1, \cdots, x^{n+2}\}$ is the affine coordinate system on $\mathbb{R}^{n+2}$.

Definition 2.12. An immersion $f : \mathbf{M} \longrightarrow \mathbb{R}^{n+2}$ is called a centro-affine immersion of codimension two if there exists a vector field ξ along f (at least locally) such that

$$T_{f(p)}\mathbb{R}^{n+2} = f_*(T_p\mathbf{M}) \oplus R\{\eta_{f(p)}\} \oplus R\{\xi_{f(p)}\}, \tag{2.65}$$

where $R\{\eta\}$ and $R\{\xi\}$ are 1-dimensional subspaces spanned by η and ξ, respectively. The vector field ξ is called the transversal vector field.

As a result of this decomposition the vector fields $D_X f_* Y$ and $D_X \xi$, where X and Y are vector fields on $\mathbf{M}$, are written as follows

$$D_X f_* Y = f_*(\nabla_X Y) + T(X,Y)\eta + h(X,Y)\xi, \tag{2.66}$$
$$D_X \xi = -f_*(SX) + \mu(X)\eta + \tau(X)\xi, \tag{2.67}$$

where ∇ is a torsion free affine connection, T, h are symmetric $(0,2)$-tensor fields, μ, τ are 1-forms and S is a tensor field of type $(1,1)$ on $\mathbf{M}$ called the shape operator. The affine connection ∇ is called the induced connection, h is called the second fundamental form and τ is called the transversal connection form.

Since the connection D is flat the fundamental equations are

$$\begin{aligned} R(X,Y)Z &= h(Y,Z)SX - h(X,Z)SY \\ &\quad - T(Y,Z)X + T(X,Z)Y, && (2.68) \\ \left(\nabla_X T\right)(Y,Z) + \mu(X)h(Y,Z) &= \left(\nabla_Y T\right)(X,Z) + \mu(Y)h(X,Z), && (2.69) \\ \left(\nabla_X h\right)(Y,Z) + \tau(X)h(Y,Z) &= \left(\nabla_Y h\right)(X,Z) + \tau(Y)h(X,Z), && (2.70) \\ \left(\nabla_X S\right)(Y) - \tau(X)SY + \mu(X)Y &= \left(\nabla_Y S\right)(X) - \tau(Y)SX + \mu(Y)X, && (2.71) \\ T(X,SY) - T(Y,SX) &= \left(\nabla_X \mu\right)(Y) - \left(\nabla_Y \mu\right)(X) + \tau(Y)\mu(X) \\ &\quad - \tau(X)\mu(Y), && (2.72) \\ h(X,SY) - h(Y,SX) &= \left(\nabla_X \tau\right)(Y) - \left(\nabla_Y \tau\right)(X). && (2.73) \end{aligned}$$

The equation (2.68) is called the Gauss equation, (2.69), (2.70), (2.71) are called the Codazzi equation for T, the Codazzi equation for h and the Codazzi equation for S, respectively. The equations (2.72) and (2.73) are called the Ricci equations.
Now, to see the impact of the change in the transversal vector field and the immersion.

Lemma 2.1. *[13] Let $f : \mathbf{M} \longrightarrow \mathbb{R}^{n+2}$ be a centro-affine immersion of codimension two with transversal vector field ξ. Let λ be a non-zero scalar function, a be a scalar function and U be a tangent vector field on $\mathbf{M}$. Suppose the transversal vector field ξ is changed to*

$$\lambda\tilde{\xi} = \xi + a\eta + f_*U. \tag{2.74}$$

Then, the change in the induced connection, the affine fundamental form and the transversal connection form are as follows:

$$\begin{aligned} \tilde{\nabla}_X Y &= \nabla_X Y - h(X,Y)U, && (2.75) \\ \tilde{h}(X,Y) &= \lambda h(X,Y), && (2.76) \\ \tilde{\tau}(X) &= \tau(X) - X(log\lambda) + h(X,U). && (2.77) \end{aligned}$$

Proof. Since f is a centro-affine immersion of codimension two

$$D_X f_* Y = f_*(\nabla_X Y) + T(X,Y)\eta + h(X,Y)\xi.$$

Then, from equation (2.74)

$$D_X f_* Y = f_*(\nabla_X Y - h(X,Y)U) + \{T(X,Y) - ah(X,Y)\}\eta + \lambda h(X,Y)\tilde{\xi}. \tag{2.78}$$

On the other hand,

$$D_X f_* Y = f_*(\tilde{\nabla}_X Y) + \tilde{T}(X,Y)\eta + \tilde{h}(X,Y)\tilde{\xi}. \tag{2.79}$$

Then, by comparing equations (2.78) and (2.79)

$$\tilde{\nabla}_X Y = \nabla_X Y - h(X,Y)U, \tag{2.80}$$
$$\tilde{h}(X,Y) = \lambda h(X,Y). \tag{2.81}$$

Similarly, the other equation is obtained by considering $D_X \xi$. □

Immersion f is said to be non-degenerate if h is non-degenerate. If h is non-degenerate, we can choose a transversal vector field ξ such that $\tau = 0$ because of equation (2.77). As in case of codimension one, $\{f, \xi\}$ is said to be equiaffine if τ vanishes. In this case, from (2.70) we get $(\mathbf{M}, \nabla, h)$ is a statistical manifold. Then, we say that the statistical manifold is realized in $\mathbb{R}^{n+2}$ by centro-affine equiaffine immersion of codimension two.

Lemma 2.2. *[13] Let $f : \mathbf{M} \longrightarrow \mathbb{R}^{n+2}$ be a centro-affine immersion of codimension two with transversal vector field ξ. Let σ be a positive function on $\mathbf{M}$, change the immersion f by $\hat{f} = \sigma f$. Then, the induced connections, the affine fundamental forms and the transversal connection forms of (f, ξ) and $(\hat{f}, \xi)$ are related as:*

$$\begin{aligned}
\hat{\nabla}_X Y &= \nabla_X Y + d(log\sigma)(Y)X + d(log\sigma)(X)Y, \\
\hat{h}(X,Y) &= \sigma h(X,Y), \\
\hat{\tau}(X) &= \tau(X).
\end{aligned} \tag{2.82}$$

Proof. Recall, $\hat{f}_*(Y) = Y(\sigma)f + \sigma f_* Y$. Then,

$$\begin{aligned}
D_X \hat{f}_*(Y) &= D_X(Y(\sigma)f + \sigma f_* Y) \\
&= X(Y(\sigma))f + X(\sigma)f_* Y + Y(\sigma)f_* X \\
&\quad + \sigma\{f_*(\nabla_X Y + T(X,Y)\eta + h(X,Y)\xi)\}.
\end{aligned} \tag{2.83}$$

On the other hand

$$\begin{aligned}
D_X \hat{f}_*(Y) &= \hat{f}_*(\hat{\nabla}_X Y) + \hat{T}(X,Y)\eta + \hat{h}(X,Y)\xi \\
&= (\hat{\nabla}_X Y)(\sigma)f + \sigma f_*(\hat{\nabla}_X Y) + \hat{T}(X,Y)\eta + \hat{h}(X,Y)\xi.
\end{aligned} \tag{2.84}$$

Comparing equations (2.83) and (2.84)

$$\begin{aligned} \hat{\nabla}_X Y &= \nabla_X Y + d(log\sigma)(Y)X + d(log\sigma)(X)Y, \\ \hat{h}(X,Y) &= \sigma h(X,Y). \end{aligned}$$

Similary, the other equation is obtained by considering $D_X\xi$. □

Remark 2.4. Note that from the equation (2.82) that (f,ξ) is equiaffine if and only if $(\hat{f},\xi)$ is equiaffine.

As a consequence of the above lemmas (2.1) and (2.2) we have the following propositions.

Proposition 2.3. *[14] Let $(\mathbf{M},\nabla,h)$ be a statistical manifold realized in $\mathbb{R}^{n+2}$ by a centro-affine immersion $\{f,\xi\}$. Take a transversal vector field $\tilde{\xi} = \lambda^{-1}\{\xi + a\eta + \lambda^{-1} grad_h\phi\}$, where a is a function, ϕ a positive function on $\mathbf{M}$ and $grad_h\phi$ is the gradient vector field of ϕ with respect to h. Then, the statistical manifold $(\mathbf{M},\tilde{\nabla},\tilde{h})$ realized by $\{f,\tilde{\xi}\}$ is 1-conformally equivalent to $(\mathbf{M},\nabla,h)$.*

Proof. Proof follows from lemma (2.1) with $U = grad_h\phi$. □

Also,

Proposition 2.4. *[14] Let $(\mathbf{M},\nabla,h)$ be a statistical manifold realized in $\mathbb{R}^{n+2}$ by a centro-affine immersion $\{f,\xi\}$. Change the immersion f by $\hat{f} = \sigma f$, where σ is a positive function on $\mathbf{M}$. Then, the statistical manifold $(\mathbf{M},\hat{\nabla},\hat{h})$ realized by $\{\hat{f},\xi\}$ is (-1)-conformally equivalent to $(\mathbf{M},\nabla,h)$.*

Proof. Proof is immediate from lemma (2.2). □

In [14], Matsuzoe obtained a necessary and sufficient condition for a statistical manifold to be realized by a nondegenerate equiaffine centro-affine immersion of codimension two.

Theorem 2.12. *[14]*

Let $f : \mathbf{M} \longrightarrow \mathbb{R}^{n+2}$ be a nondegenerate equiaffine centro-affine immersion with induced connection ∇ and affine fundamental form h. Then, $(\mathbf{M},\nabla,h)$ is a conformally-projectively flat statistical manifold. Conversely, suppose that $(\mathbf{M},\nabla,h)$ is simply connected conformally-projectively flat statistical manifold of dimension n. Then, there exists a nondegenerate equiaffine centroaffine immersion $f : \mathbf{M} \longrightarrow \mathbb{R}^{n+2}$ with the induced connection ∇ and the affine fundamental from h.

2.3.1 Centro-Affine Immersion into Dually Flat Statistical Manifolds

In this subsection, we define the centro-affine immersion into a dually flat statistical manifold of codimension two, which is a generalization of the centro-affine immersion of codimension two [9].

Definition 2.13. Let $(\tilde{\mathbf{M}}, \tilde{\nabla}, \overline{\tilde{\nabla}}, \tilde{g})$ be a dually flat statistical manifold of dimension $(n+2)$ and $\mathbf{M}$ be an n-dimensional manifold. Let $\gamma = \sum_{i=1}^{n+2} \theta^i \frac{\partial}{\partial \theta_i}$ be the radial vector field of $\tilde{\mathbf{M}}$ with respect to the affine coordinate $[\theta^i]$ of $\tilde{\nabla}$. An immersion $f : \mathbf{M} \longrightarrow \tilde{\mathbf{M}}$ is called a centro-affine immersion of codimension two if there exists a unit normal vector field ξ such that

$$T_{f(p)}(\tilde{\mathbf{M}}) = f_*(T_p(\mathbf{M})) + span\{\xi\} + span\{\gamma\}.$$

In this case the Gauss and the Weingarten formulae are

$$\begin{aligned}
\tilde{\nabla}_X f_*(Y) &= f_*(\nabla_X Y) + h(X,Y)\xi + T(X,Y)\gamma, && (2.85)\\
\tilde{\nabla}_X \xi &= -f_*(SX) + \tau(X)\xi + \sigma(X)\gamma, && (2.86)\\
\tilde{\nabla}_X \gamma &= -f_*(X), && (2.87)
\end{aligned}$$

where S is the affine shape operator, $\tau(X)$ and $\sigma(X)$ are the transversal connection forms and

$$h, T : T_p(\mathbf{M}) \times T_p(\mathbf{M}) \longrightarrow \mathbf{R}$$

are the second fundamental forms. The Gauss and the Weingarten formulae for the dual connection can also be written similarly.

Let $f : \mathbf{M} \longrightarrow \tilde{\mathbf{M}}$ be a centro-affine immersion of codimension two. Then, f is said to be non-degenerate if the second fundamental form h is non-degenerate and f is equiaffine if the transversal connection form τ is zero.

Remark 2.5. From the fundamental equations of centro-affine immersion of codimension two we can show that $(\mathbf{M}, \nabla, h)$ and $(\mathbf{M}, \overline{\nabla}, \overline{h})$ are statistical manifolds for non-degenerate equiaffine, centro-affine immersion.

Now, we have

Theorem 2.13. *Let* $(\tilde{\mathbf{M}}, \tilde{\nabla}, \overline{\tilde{\nabla}}, \tilde{g})$ *be a dually flat statistical manifold of dimension* $(n+2)$ *and* $\mathbf{M}$ *be a manifold of dimension* n*. If* $f : \mathbf{M} \longrightarrow \tilde{\mathbf{M}}$ *is a non-degenerate centro-affine, equiaffine immersion of codimension two, then* $(\mathbf{M}, \nabla, h)$ *and* $(\mathbf{M}, \overline{\nabla}, \overline{h})$ *are dual to each other if and only if* $S = \overline{S} = \lambda I$ *for some constant* λ*.*

Proof. Proof is same as that of the theorem (2.4) □

Now, we show that for a non-degenerate centro-affine, equiaffine immersion of codimension two into a dually flat statistical manifold the inherited structure $(\mathbf{M}, \nabla, h)$ is a conformally-projctively flat statistical manifold.

Theorem 2.14. *Let* $(\tilde{\mathbf{M}}, \tilde{\nabla}, \overline{\tilde{\nabla}}, \tilde{g})$ *be a dually flat connected statistical manifold of dimension* $(n+2)$ *and* $\mathbf{M}$ *be a manifold of dimension* n*. Let* $f : \mathbf{M} \longrightarrow \tilde{\mathbf{M}}$ *be a non-degenerate centro-affine, equiaffine immersion of codimension two. Then, the inherited statistical manifold* $(\mathbf{M}, \nabla, h)$ *is conformally-projectively flat.*

Proof. Let $f : \mathbf{M} \longrightarrow \tilde{\mathbf{M}}$ be a non-degenerate, centro-affine equiaffine immersion of codimension two. For $p \in \mathbf{M}$ the affine coordinate of $f(p)$ in $\tilde{\mathbf{M}}$ with respect to $\tilde{\nabla}$ is denoted by $\theta = [\theta^i]$.

Let $\eta = \sum_{i=1}^{n+2} \theta^i \frac{\partial}{\partial \theta_i}$ be the radial vector field of f in $\tilde{\mathbf{M}}$, ξ be the normal vector field of f in $\tilde{\mathbf{M}}$. Then,

$$T_{f(p)}(\tilde{\mathbf{M}}) = f_*(T_p(\mathbf{M})) + span\{\xi_p\} + span\{\eta_p\}.$$

Also,

$$\begin{aligned} \tilde{\nabla}_X f_*(Y) &= f_*(\nabla_X Y) + h(X,Y)\xi + T(X,Y)\eta. \\ \tilde{\nabla}_X \xi &= -f_*(SX) + \tau(X)\xi + \sigma(X)\eta. \end{aligned}$$

For a positive function $\psi : \mathbf{M} \longrightarrow \mathbf{R}$, define $J(p) = \psi(p)\theta(f(p))$. Then, $J : \mathbf{M} \longrightarrow \tilde{\mathbf{M}}$ is a centro-affine immersion of codimension two with the radial vector field η and the unit normal vector field ξ. Then,

$$T_{J(p)}(\tilde{\mathbf{M}}) = J_*(T_p(\mathbf{M})) + span\{\xi_p\} + span\{\eta_p\}.$$

Also,

$$\begin{aligned} \tilde{\nabla}_X J_*(Y) &= J_*(\hat{\nabla}_X Y) + \hat{h}(X,Y)\xi + \hat{T}(X,Y)\eta, \\ \tilde{\nabla}_X \xi &= -J_*(\hat{S}X) + \hat{\tau}(X)\xi + \hat{\sigma}(X)\eta, \end{aligned}$$

where $\hat{\nabla}$ is the induced connection with respect to the immersion J. The tensors $\hat{h}, \hat{T}$ are the second fundamental forms, $\hat{S}$ the shape operator and $\hat{\tau}, \hat{\sigma}$ are the transversal connection forms.
Then, (∇, h, τ) and $(\hat{\nabla}, \hat{h}, \hat{\tau})$ are related by

$$\begin{aligned} \hat{\nabla}_X Y &= \nabla_X Y + d(log\psi)(Y)X + d(log\psi)(X)Y. \qquad (2.88) \\ \hat{h} &= \psi h. \qquad (2.89) \\ \hat{\tau} &= \tau. \qquad (2.90) \end{aligned}$$

For each $p \in \mathbf{M}$, choose ψ in some neighborhood $\mathbf{U}_p$ of p such that the coefficient of η for the immersion $J = \psi f$ is zero. Take a transversal vector field $\xi^{'}$ on $\mathbf{U}_p$ with ξ restricted to $\mathbf{U}_p$ is $\xi^{'}$. Since $J_*(T_x\mathbf{M}), \eta_{J(x)}$ and ξ_x are linearly independent for $x \in \mathbf{U}_p$ there exists a positive function ϕ, a function a and a tangent vector field V on $\mathbf{U}_p$ such that $\phi\xi^{'}_x = \xi_x + a\eta_{J(x)} + J_*V$.
Now, consider the immersion $(J, \xi^{'}, \eta) : \mathbf{U}_p \longrightarrow \tilde{\mathbf{M}}$ then $T^{'} = 0$. Also,

$$\begin{aligned} \nabla^{'}_X Y &= \hat{\nabla}_X Y - \hat{h}(X,Y)V, \qquad (2.91) \\ h^{'} &= \phi\hat{h}, \qquad (2.92) \\ \tau^{'}(X) &= \hat{\tau}(X) - X(log\phi) + \hat{h}(X,V). \qquad (2.93) \end{aligned}$$

Since $\xi^{'}$ is parallel around p, $\tilde{\nabla}_X\xi^{'} = 0$ which implies $\mu^{'} = S^{'} = \tau^{'} = 0$. Then, by the Gauss equation $R^{\nabla^{'}}(X,Y)Z = h^{'}(Y,Z)S^{'}X - h^{'}(X,Z)S^{'}Y - T^{'}(Y,Z)X + T^{'}(X,Z)Y$, where X, Y, Z are vector fields on $\mathbf{U}_p$ we get $\nabla^{'}$ is flat. That is, $(\mathbf{U}_p, \nabla^{'}, h^{'})$ is a flat statistical manifold.

Since ξ_x and $\xi^{'}_x$ are equiaffine from (2.93)

$$\hat{h}(X,V) = X(log\phi).$$

Then,

$$V = \frac{1}{\phi\psi} grad_h\phi. \qquad (2.94)$$

From equations (2.88) to (2.94)

$$\begin{aligned} h^{'} &= \phi\psi h. \\ h(\nabla^{'}_X Y, Z) &= h(\nabla_X Y, Z) - d(log\phi)(Z)h(X,Y) + d(log\psi)(X)h(Y,Z) \\ &\quad + d(log\psi)(Y)h(X,Z). \end{aligned}$$

Hence, $(\mathbf{M}, \nabla, h)$ is conformally-projectively flat. $\square$

2.4 Affine Immersions of General Codimension

In this section, we first discuss the affine fundamental form and relation between curvature tensors for affine immersions of general codimension [2]. Transversal volume element map is defined for equiaffine statistical immersion of general codimension and certain properties also proved. Also, we give a detailed proof of the sufficient condition given by Matsuzoe et al. [16], for a statistical submanifold of a dually flat statistical manifold to be equiaffine.

Let $\mathbf{M}$ and $\tilde{\mathbf{M}}$ be two smooth manifolds of dimension n and $(n+r)$ with torsion-free affine connections ∇ and $\tilde{\nabla}$ respectively. An immersion $f : \mathbf{M} \longrightarrow \tilde{\mathbf{M}}$ is called an affine immersion if there exists an r-dimensional smooth distribution $\mathcal{N}$ along f which assigns to every point $p \in \mathbf{M}$ a subspace $\mathcal{N}_p$ of $T_{f(p)}(\tilde{\mathbf{M}})$ such that the following equations hold

$$
\begin{aligned}
T_{f(p)}(\tilde{\mathbf{M}}) &= f_*(T_p(\mathbf{M})) \bigoplus \mathcal{N}_p, && (2.95)\\
(\tilde{\nabla}_X f_* Y)_p &= (f_*(\nabla_X Y))_p + \alpha(X,Y)_p, && (2.96)
\end{aligned}
$$

where $\alpha(X,Y)_p \in \mathcal{N}_p$ at each point p in $\mathbf{M}$ and X, Y are in $\mathcal{X}(\mathbf{M})$. The $(0,2)$-tensor $\alpha(X,Y)$ is called the affine fundamental form, the choice of which is not unique in general. The affine fundamental form defines a mapping

$$\alpha_p : T_p(\mathbf{M}) \times T_p(\mathbf{M}) \longrightarrow \mathcal{N}_p, \text{ for } p \in \mathbf{M}.$$

Rank of this map is called the rank of the affine fundamental form. Also, the Weingarten decomposition is

$$\tilde{\nabla}_X \xi = -f_*(S_\xi X) + \nabla^\perp_X \xi, \tag{2.97}$$

where ξ is the vector field on the transversal bundle $\mathcal{N}$, $f_*(S_\xi X) \in T\mathbf{M}$ and $\nabla^\perp_X \xi \in \mathcal{N}$. This defines the shape operator S_ξ and the transversal connection $\nabla^\perp$.

In the general codimension case also we have equations relating the curvature tensors [2]. Let the prefixes tan and nor denote the tangential and normal components of the vector in $T_{f(p)}(\tilde{\mathbf{M}})$ according to the decomposition (2.95).

$$
\begin{aligned}
tan\tilde{R}(X,Y)Z &= R(X,Y)Z + S_{\alpha(X,Z)}Y - S_{\alpha(Y,Z)}X, && (2.98)\\
nor\tilde{R}(X,Y)Z &= (\nabla_X \alpha)(Y,Z) - (\nabla_Y \alpha)(X,Z), && (2.99)\\
tan\tilde{R}(X,Y)\xi &= (\nabla_Y S)_\xi(X) - (\nabla_X S)_\xi Y, && (2.100)\\
nor\tilde{R}(X,Y)\xi &= \alpha(S_\xi X, Y) - \alpha(X, S_\xi Y) + R^\perp(X,Y)\xi, && (2.101)
\end{aligned}
$$

where X, Y, Z are in $\mathcal{X}(\mathbf{M})$ and $R^{\perp}(X, Y)$ is the curvature tensor of the transversal connection. The equation (2.98) is called the Gauss equation, (2.99) and (2.100) are called the Codazzi equations. The equation (2.101) is called the Ricci equation.

Now, we consider immersions of general codimension in the case of statistical manifolds. Let $(\mathbf{M}, \nabla', \overline{\nabla}', g)$ and $(\tilde{\mathbf{M}}, \tilde{\nabla}, \overline{\tilde{\nabla}}, \tilde{g})$ be statistical manifolds of dimensions n and $(n+r)$, respectively. Let $f : \mathbf{M} \longrightarrow \tilde{\mathbf{M}}$ be a statistical immersion, since $(\tilde{\mathbf{M}}, \tilde{\nabla}, \tilde{g})$ is a semi-Riemannian manifold we can choose $\mathcal{N} = T\mathbf{M}^{\perp}$. The connection ∇ on $\mathbf{M}$ and $T\mathbf{M}^{\perp}$ valued symmetric $(0, 2)$-tensor field $\alpha(X, Y)$ on $\mathbf{M}$ are defined by

$$\tilde{\nabla}_X f_* Y = f_*(\nabla_X Y) + \alpha(X, Y) \quad \text{for all} \quad X, Y \in \mathcal{X}(\mathbf{M}),$$

where $f_*(\nabla_X Y)$ is the $f_*T\mathbf{M}$- component of $\tilde{\nabla}_X f_* Y$ and $\alpha(X, Y)$ is the $T\mathbf{M}^{\perp}$- component of $\tilde{\nabla}_X f_* Y$. Let $\{\xi_1, \xi_2 ... \xi_r\}$ denote the frame for $T\mathbf{M}^{\perp}$ then

$$\begin{aligned} \alpha(X, Y) &= \sum_{i=1}^{r} \alpha_i(X, Y)\xi_i, \\ \nabla_X^{\perp}\xi &= \sum_{j=1}^{r} \tau_{ij}(X)\xi_j, \end{aligned}$$

where $\nabla^{\perp}$ is the transversal connection form.

Since f is a statistical immersion observe that ∇ coincides with ∇'. Also, for dual connection $\overline{\tilde{\nabla}}$ we have

1. $\overline{\tilde{\nabla}}_X f_* Y = f_*(\overline{\nabla}_X Y) + \overline{\alpha}(X, Y) \;\; \forall \;\; X, Y \in \mathcal{X}(\mathbf{M})$.

2. $\overline{\tilde{\nabla}}_X \xi = -f_*(\overline{S}_{\xi} X) + \overline{\nabla}_X^{\perp}\xi$.

3. $\overline{\alpha}(X, Y) = \sum_{i=1}^{r} \overline{\alpha}_i(X, Y)\xi_i$.

4. $\overline{\nabla}_X^{\perp}\xi = \sum_{j=1}^{r} \overline{\tau}_{ij}(X)\xi_j$.

Note that, in this case $\overline{\nabla}'$ coincides with $\overline{\nabla}$.

Let $(\tilde{\mathbf{M}}, \tilde{\nabla}, \overline{\tilde{\nabla}}, \tilde{g})$ be an $(n+r)$-dimensional statistical manifold and $\mathbf{M}$ be an n-dimensional smooth manifold. Let $f : \mathbf{M} \longrightarrow \tilde{\mathbf{M}}$ be an immersion of codimension r. We have the induced metric $f^*\tilde{g}$ and the induced connection ∇ on $\mathbf{M}$ defined by

$$f^*\tilde{g}(\nabla_X Y, Z) = \tilde{g}(\tilde{\nabla}_X f_* Y, f_* Z).$$

Then, $(\mathbf{M}, \nabla, f^*\tilde{g})$ is a statistical manifold and the induced connections ∇ and $\overline{\nabla}$ are dual with respect to $f^*\tilde{g}$.

Definition 2.14. Let $\tilde{\theta}$ be the volume element on $\tilde{\mathbf{M}}$ and $\theta^{\perp}$ be the volume element on $T\mathbf{M}^{\perp}$. Then, the volume element on $\mathbf{M}$ is defined as

$$\theta(X_1, X_2 ... X_n) = \frac{\tilde{\theta}(f_*X_1, f_*X_2, ... f_*X_n, \xi_1, \xi_2 ... \xi_r)}{\theta^{\perp}(\xi_1, \xi_2 ... \xi_r)}, \tag{2.102}$$

where $\{\xi_1, \xi_2 ... \xi_r\}$ is a frame of $T\mathbf{M}^{\perp}$. We call θ the induced volume element for $(T\mathbf{M}^{\perp}, \theta^{\perp})$.

Remark 2.6. If $\nabla\theta = 0$, then f is said to be an equiaffine immersion of $(\mathbf{M}, \nabla)$ into $(\tilde{\mathbf{M}}, \tilde{\nabla})$ and we say that $(T\mathbf{M}^{\perp}, \theta^{\perp})$ is equiaffine. f is said to be non-degenerate if α is non-degenerate.

Proposition 2.5. *[15] Let $(\tilde{\mathbf{M}}, \tilde{\nabla}, \tilde{\theta})$ be an equiaffine manifold, then the condition $\nabla\theta = 0$ and $\nabla^{\perp}\theta^{\perp} = 0$ are equivalent.*

Proof. Take an arbitrary point $x \in \mathbf{M}$ and a local frame $\{\xi_1, \xi_2 ... \xi_r\}$ of $T\mathbf{M}^{\perp}$ on a neighborhood U of x with $\theta^{\perp}(\xi_1, \xi_2 ... \xi_r) = 1$. Then,

$(\tilde{\nabla}_X\tilde{\theta})(f_*X_1, ..., f_*X_n, \xi_1, ..., \xi_r)$
$= X(\tilde{\theta}(f_*X_1, ..., f_*X_n, \xi_1, ..., \xi_r)) - \sum_{i=1}^{n} \tilde{\theta}(f_*X_1, ... \tilde{\nabla}_X f_*X_i, ..., f_*X_n, \xi_1, ..., \xi_r)$
$- \sum_{i=1}^{r} \tilde{\theta}(f_*X_1, ..., f_*X_n, \xi_1, ..., \tilde{\nabla}_X\xi_i, ..., \xi_r)$
$= X(\tilde{\theta}(f_*X_1, ..., f_*X_n, \xi_1, ..., \xi_r)) - \sum_{i=1}^{n} \tilde{\theta}(f_*X_1, ... f_*(\nabla_X X_i), ..., f_*X_n, \xi_1, ..., \xi_r)$
$- \sum_{i=1}^{r} \tilde{\theta}(f_*X_1, ..., f_*X_n, \xi_1, ..., \tilde{\nabla}_X\xi_i, ..., \xi_r)$
$= X(\theta(X_1, ..., X_n)) - \sum_{i=1}^{n} \theta(X_1, ... \nabla_X X_i, ..., X_n) - \sum_{i=1}^{r} \tau_{ii}(X)\theta(X_1, ..., X_n)$
$= (\nabla_X\theta)(X_1, ... \nabla_X X_i, ..., X_n)$ - $\sum_{i=1}^{r} \tau_{ii}(X)\theta(X_1, ..., X_n)$,

for every $X_1, ..., X_1$ in $\mathcal{X}(\mathbf{M})$. Since $(\tilde{\mathbf{M}}, \tilde{\nabla}, \tilde{\theta})$ is equiaffine

$$(\nabla_X\theta)(X_1, ... \nabla_X X_i, ..., X_n) = \sum_{i=1}^{r} \tau_{ii}(X)\theta(X_1, ..., X_n).$$

Thus, $\nabla\theta = 0$ is equivalent to $\sum_{i=1}^{r}\tau_{ii}(X) = 0$. Now,

$$\begin{aligned}
(\nabla^{\perp}_{X}\theta^{\perp})(\xi_1,...,\xi_r) &= X(\theta^{\perp})(\xi_1,...,\xi_r)) - \sum_{i=1}^{r}\theta^{\perp})(\xi_1,...,\nabla^{\perp}_{X}\xi_i,...,\xi_r) \\
&= -\sum_{i=1}^{r}\theta^{\perp})(\xi_1,...,\nabla^{\perp}_{X}\xi_i,...,\xi_r) \\
&= -\sum_{i=1}^{r}\tau_{ii}(X).
\end{aligned}$$

Thus $\nabla\theta = 0$ is equivalent to $\nabla^{\perp}\theta^{\perp} = 0$. $\square$

Now, we define the transversal volume element map in the case of statistical immersion of general codimension and obtain certain properties of it. Let $(\tilde{\mathbf{M}}, \tilde{\nabla}, \tilde{g})$ be an $(n+r)$-dimensional statistical manifold with equiaffine structure $(\tilde{\nabla}, \tilde{\theta})$ and $(\mathbf{M}, \nabla, g)$ be a statistical manifold of dimension n. Let $f : \mathbf{M} \longrightarrow \tilde{\mathbf{M}}$ be an equiaffine statistical immersion of codimension r.

Definition 2.15. The map $\nu : \mathbf{M} \longrightarrow \bigwedge^{r}(\mathbf{T}\tilde{\mathbf{M}})$ defined by $ker\nu_p = f_*(T_p\mathbf{M})$ and $\nu_p \mid_{T_p\mathbf{M}^{\perp}\times T_p\mathbf{M}^{\perp}....\times T_p\mathbf{M}^{\perp}} = \theta^{\perp}_{p}$, for $p \in \mathbf{M}$, is called the transversal volume element map for $(\mathbf{TM}^{\perp}, \theta^{\perp})$.

Lemma 2.3. *For every $X, Y \in T_p\mathbf{M}$ and every frame $(\xi_1, \xi_2, ...\xi_r)$ of $\mathbf{TM}^{\perp}$ the following relations hold*

$$\begin{aligned}
\nu_*(Y)(\xi_1, \xi_2...\xi_r) &= 0. \\
\nu_*(Y)(f_*X, ...) &= -\nu(\alpha(X, Y), ...).
\end{aligned}$$

Proof. Take a frame field $(\tilde{\xi_1}, \tilde{\xi_2}, ...\tilde{\xi_r})$ of $\mathbf{TM}^{\perp}$ on a neighborhood of p such that $(\tilde{\xi_i})_p = \xi_i$, for $i = 1, ...r$ and $\nu(\tilde{\xi_1}, \tilde{\xi_2}, ...\tilde{\xi_r})$ is constant. By differentiating $\nu(\tilde{\xi_1}, \tilde{\xi_2}, ...\tilde{\xi_r})$ in the direction Y

$$\begin{aligned}
Y(\nu(\tilde{\xi_1}, \tilde{\xi_2}, ...\tilde{\xi_r})) &= (\tilde{\nabla}_Y\nu)(\xi_1, \xi_2, ...\xi_r) + \sum_{i=1}^{r}\nu(\xi_1, ..., \tilde{\nabla}_Y\tilde{\xi_i}, ..., \xi_r) \\
&= \nu_*(Y)(\xi_1, \xi_2, ...\xi_r) + \sum_{i=1}^{r}\nu(\xi_1, ..., \nabla^{\perp}_{Y}\tilde{\xi_i}, ..., \xi_r) \\
&= \nu_*(Y)(\xi_1, \xi_2, ...\xi_r) + \sum_{i=1}^{r}\nu(\xi_1, ..., \sum_{j=1}^{r}\tau_{ij}(Y)\xi_j, ..., \xi_r).
\end{aligned}$$

This implies

$$\nu_*(Y)(\xi_1, \xi_2, ...\xi_r) + \sum_{i=1}^{r} \tau_{ii}(Y)\nu(\xi_1, ..., \xi_i, ..., \xi_r) = 0.$$

Since $(\mathbf{TM}^\perp, \theta^\perp)$ is equiaffine

$$\nu_*(Y)(\xi_1, \xi_2 ... \xi_r) = 0.$$

Similarly, we can prove the other equation. □

Proposition 2.6. *If f is non-degenerate, then ν is an immersion.*

Proof. Take a point $p \in \mathbf{M}$ and $Y \in T_p(\mathbf{M})$ with $Y \neq 0$. Since f is non-degenerate, there exists $X \in T_p(\mathbf{M})$ with $\alpha(X, Y) \neq 0$, then from the above lemma $\nu_*(Y)(f_*X, ...) \neq 0$. In particular, $\nu_*(Y) \neq 0$ and this implies ν is an immersion. □

In [16], Matsuzoe et al. discuss about the equiaffine structure on a dually flat statistical manifold in the case of general codimension. Let $(\tilde{\mathbf{M}}, \tilde{\nabla}, \overline{\tilde{\nabla}}, \tilde{g})$ be a dually flat statistical manifold of dimension n. Let $\mathbf{M}$ be a manifold of dimension m ($m < n$) embedded in $\tilde{\mathbf{M}}$. Then, $(\mathbf{M}, \nabla, \overline{\nabla}, g)$ is a statistical manifold, where $g, \nabla, \overline{\nabla}$ are induced from $\tilde{g}, \tilde{\nabla}, \overline{\tilde{\nabla}}$ respectively. Let $\{\xi_\lambda\}$, for $\lambda = m+1, ..., n$ be a basis for the normal space of $\mathbf{M}$, then

$$\begin{aligned} \tilde{\nabla}_X Y &= \nabla_X Y + \sum_{\lambda=m+1}^{n} \alpha^\lambda(X, Y)\xi_\lambda. \\ \tilde{\nabla}_X \xi_\lambda &= -S_\lambda(X) + \sum_{k=m+1}^{n} \tau_\lambda^k(X)\xi_k. \end{aligned}$$

Since $\tilde{\nabla}$ is dually flat the fundamental equations are

1. $R^\nabla(X, Y)Z = \sum_{\lambda=m+1}^{n}[\alpha^\lambda(Y, Z)S_\lambda X - \alpha^\lambda(X, Z)S_\lambda Y]$.
2. $(\nabla_X \alpha^\lambda)(Y, Z) + \sum_{k=m+1}^{n} \tau_k^\lambda(X)\alpha^k(Y, Z) = (\nabla_Y \alpha^\lambda)(X, Z) + \sum_{k=m+1}^{n} \tau_k^\lambda(Y)\alpha^k(X, Z)$.
3. $(\nabla_X S_\lambda)(Y) + \sum_{k=m+1}^{n} \tau_\lambda^k(Y)S_k X = (\nabla_Y S_\lambda)(X) + \sum_{k=m+1}^{n} \tau_\lambda^k(X)S_k Y$.
4. $\alpha^k(X, S_\lambda Y) - (\nabla_X \tau_\lambda^k)(Y) - \sum_{\nu=m+1}^{n} \tau_\lambda^\nu(Y)\tau_\nu^k(X) = \alpha^k(Y, S_\lambda X) - (\nabla_Y \tau_\lambda^k)(X) - \sum_{\nu=m+1}^{n} \tau_\lambda^\nu(X)\tau_\nu^k(Y)$.

From (1) and (4)

$$\begin{aligned} Ric(Y, Z) &= \alpha^\lambda(Y, Z) tr S_\lambda - \alpha^\lambda(S_\lambda(Y), Z), \qquad (2.103) \\ \alpha^\lambda(X, S_\lambda(Y)) - (\nabla_X \tau_\lambda^\lambda)(Y) &= \alpha^\lambda(Y, S_\lambda(X)) - (\nabla_Y \tau_\lambda^\lambda)(X). \qquad (2.104) \end{aligned}$$

Proposition 2.7. *[16] The following statements are equivalent.*

1. *Ric is a symmetric tensor field.*

2. $\sum_\lambda \alpha^\lambda(S_\lambda(.), .)$ *is a symmetric tensor field.*

3. $d(\sum_\lambda \tau_\lambda^\lambda)$ = 0.

Proof. Suppose Ric is symmetric, then from the equation (2.103) we get $\alpha^\lambda(S_\lambda(Y), Z)$ is symmetric. So (2) holds. Assume $\sum_\lambda \alpha^\lambda(S_\lambda(.), .)$ is a symmetric tensor field. Then, from the equation (2.104) we get $(\nabla_X\tau_\lambda^\lambda)(Y) - (\nabla_Y\tau_\lambda^\lambda)(X) = 0$. This implies (3). Let $d(\sum_\lambda \tau_\lambda^\lambda)$ = 0. Then, from the equation (2.104) we get $\alpha^\lambda(X, S_\lambda(Y))$ is symmetric. Thus from the equation (2.103) Ric is a symmetric tensor field. □

Let $(u^1, u^2, ...u^n)$ be a local coordinate system on $\tilde{M}$ such that the coordinate on $\mathbf{M}$ is $\{(u^1, u^2, ...u^m) \mid u^{m+1} = u^{m+2}.... = u^n = 0\}$, then $\tilde{g}(\partial_a, \partial_\lambda) = 0$ on $\mathbf{M}$, for $a = 1, ...m \ and \ \lambda = m + 1, ...n$ where $\{u^\lambda\}$ form a basis for the normal space of $\mathbf{M}$. Now, we have the following equations

$$\begin{aligned}
\tilde{\nabla}_{\partial i}\partial_j &= \sum_{a=1}^{m} \Gamma_{ij}^a \partial_a + \sum_{\lambda=m+1}^{n} \Gamma_{ij}^\lambda \partial_\lambda. \\
\tilde{\nabla}_{\partial \lambda}\partial_j &= \sum_{a=1}^{m} \Gamma_{i\lambda}^a \partial_a + \sum_{k=m+1}^{n} \Gamma_{i\lambda}^k \partial_k.
\end{aligned}$$

Proposition (2.7) can be written as

Proposition 2.8. *[16] The following statements are equivalent.*

1. *Ric is a symmetric tensor field.*

2. $\sum_{\lambda=m+1}^{n} \sum_{a=1}^{m} \Gamma_{i\lambda}^a \Gamma_{aj}^\lambda$ = $\sum_{\lambda=m+1}^{n} \sum_{a=1}^{m} \Gamma_{j\lambda}^a \Gamma_{ai}^\lambda$.

3. $\partial_j(\sum_{\lambda=m+1}^{n} \Gamma_{i\lambda}^\lambda)$ = $\partial_i(\sum_{\lambda=m+1}^{n} \Gamma_{j\lambda}^\lambda)$.

Let C denote the cubic form with respect to g, that is

$$C(Y, Z, X) = (\nabla_X g)(Y, Z).$$

Similarly, the cubic form with respect to $\tilde{g}$ is denoted by $\tilde{C}$. Set $\tilde{g}(\xi_\lambda, \xi_k) = \delta_{\lambda k}$, for λ, k = $m + 1, ...n$. In [16], Matsuzoe et al. proved that

$$\sum_\lambda \tau_\lambda^\lambda(X) = (-1/2) \sum_\lambda \tilde{C}(\xi_\lambda, \xi_\lambda, X),$$

that is, if $\sum_\lambda \tilde{C}(\xi_\lambda, \xi_\lambda, X) = 0$, then Ric is symmetric.

Theorem 2.15. *[16] Let* $(\tilde{\nabla}, \tilde{\omega})$ *be an equiaffine structure on* $\tilde{\mathrm{M}}$*, set*

$$\begin{aligned} \omega(X_1, X_2, ...X_m) &= \tilde{\omega}(X_1, X_2, ...X_m, \xi_{m+1}, \xi_{m+2}, ..\xi_n). \\ T(X) &= tr_g C(*, *, X). \\ \tilde{T}(X) &= tr_{\tilde{g}} \tilde{C}(*, *, X). \end{aligned}$$

If $T = \tilde{T}$ *on* M*, then* (∇, ω) *is an equiaffine structure on* M.

Proof. Consider,

$$(\nabla_X \omega)(X_1, X_2...X_n) = \sum_\lambda \tau_\lambda^\lambda(X) \omega(X_1, X_2...X_n).$$

Also,

$$\begin{aligned} (-2) \sum_\lambda \tau_\lambda^\lambda(X) &= \sum_\lambda \tilde{C}(\xi_\lambda, \xi_\lambda, X) \\ &= tr_{\tilde{g}} \tilde{C}(*, *, X) - tr_g C(*, *, X) \\ &= \tilde{T}(X) - T(X). \end{aligned}$$

Now, if $T = \tilde{T}$ on M, then (∇, ω) is an equiaffine structure on M. □

Note. The equiaffine immersion of codimension one as well as the non-degenerate equiaffine centro-affine immersion of codimension two induces statistical structures. In both the cases we obtained a torsion-free affine connection and the semi-Riemannian metric. There is a natural choice of semi-Riemannian metric that is independent of the choice of transversal vector field. However, in the case of immersions of general codimension the affine fundamental form is a $T\mathrm{M}^\perp$ valued symmetric $(0, 2)$-tensor. Defining a suitable metric on M and realizing a statistical manifold in an affine space or in a statistical manifold are open problems.

Chapter 3

Geometry of Submersions and Statistical Manifolds

Riemannian submersion is a special tool in differential geometry and it has got application in different areas such as Kaluza-Klein theory, Yang-Mills theory, supergravity and superstring theories, statistical machine learning processes, medical imaging, theory of robotics and the statistical analysis on manifolds. Riemannian submersions from a statistical viewpoint were first mentioned by Barndroff-Neilsen and Jupp [17]. O'Neill [18] defined a Riemannian submersion and obtained the fundamental equations of Riemannian submersions for Riemannian manifolds. Also in [19], O'Neill defined a semi-Riemannian submersion. Abe and Hasegawa [21] defined an affine submersion with horizontal distribution and obtained the fundamental equations. For the semi-Riemannian submersion $\pi : (\mathbf{M}, g_m) \rightarrow (\mathbf{B}, g_b)$, Abe and Hasegawa [21] obtained a necessary and sufficient condition for $(\mathbf{M}, \nabla, g_m)$ to become a statistical manifold with respect to the affine submersion with horizontal distribution $\pi : (\mathbf{M}, \nabla) \rightarrow (\mathbf{B}, \nabla^*)$. Conformal submersion and the fundamental equations of conformal submersion were also studied by many researchers, see [46], [47] for example. Harmonic morphisms between Riemannian manifolds of arbitrary dimensions and horizontally conformal submersions were introduced by Fuglede [48] and Ishihara [49] independently. Harmonic morphisms are nothing but harmonic and horizontally conformal maps. Their study focuses on the conformality relation between metrics on the Riemannian manifolds and the Levi-Civita connections. Our interest is on conformal submersion between Riemannian manifolds $\mathbf{M}$ and $\mathbf{B}$ and the conformality relation between any two affine connections ∇ and ∇^* (not necessarily be the Levi-Civita connections) on $\mathbf{M}$ and $\mathbf{B}$, respectively.

In Section 3.1, we discussed certain results regarding submersion and semi-Riemannian submersion for Riemannian manifolds. In Section 3.2, affine submersion with horizontal distribution for Riemannian manifolds is considered. In Section 3.3, we define the concept of a conformal submersion with horizontal distribution for Riemannian manifolds, which is a generalization of the affine submersion with horizontal distribution. Then, a necessary condition for the existence of such a map is proved. A necessary and sufficient condition is obtained for $\pi \circ \sigma$ to be a geodesic of $\mathbf{B}$ when σ is a geodesic of $\mathbf{M}$ for a conformal submersion with horizontal distribution. Then, proved a necessary and sufficient condition for the horizontal lift of a geodesic to be geodesic. Also, we give a necessary condition for the connection on $\mathbf{B}$ to be complete when the connection on $\mathbf{M}$ is complete for a conformal submersion with horizontal distribution $\pi : \mathbf{M} \longrightarrow \mathbf{B}$. In Section 3.4, we discuss affine and conformal submersion with horizontal distribution and statistical manifolds. A statistical structure is obtained on the manifold $\mathbf{B}$ induced by the affine submersion $\pi : \mathbf{M} \longrightarrow \mathbf{B}$ with the horizontal distribution $\mathcal{H}(\mathbf{M}) = \mathcal{V}(\mathbf{M})^{\perp}$ [21]. In the case of conformal submersion with horizontal distribution we obtained a necessary and sufficient condition for $(\mathbf{M}, \nabla, g_m)$ to become a statistical manifold. Also, we prove $\pi : (\mathbf{M}, \nabla) \longrightarrow (\mathbf{B}, \nabla^*)$ is a conformal submersion with horizontal distribution if and only if $\pi : (\mathbf{M}, \overline{\nabla}) \longrightarrow (\mathbf{B}, \overline{\nabla}^*)$ is a conformal submersion with horizontal distribution [22].

3.1 Semi-Riemannian Submersion

In this section, definitions of submersion and semi-Riemannian submersion for Riemannian manifolds and certain basic results are given. Comparison of geodesics by O'Neill for semi-Riemannian submersions is also discussed [18], [20].

Let $(\mathbf{M}, g_m)$ and $(\mathbf{B}, g_b)$ be smooth Riemannian manifolds of dimension m and n ($m \geq n$), respectively.

Definition 3.1. A smooth onto map $\pi : \mathbf{M} \longrightarrow \mathbf{B}$ is called a submersion if $\pi_{*p} : T_p\mathbf{M} \longrightarrow T_{\pi(p)}\mathbf{B}$ is onto for all $p \in \mathbf{M}$.
For a submersion $\pi : \mathbf{M} \longrightarrow \mathbf{B}$, $\pi^{-1}(b)$ is a submanifold of $\mathbf{M}$ of dimension $(m - n)$ for each $b \in \mathbf{B}$. These submanifolds $\pi^{-1}(b)$ are called the fibers.

A vector field on $\mathbf{M}$ is called vertical if it is tangent to the fibers and is horizontal if it is orthogonal to the fibers. Set $\mathcal{V}(\mathbf{M})_p = ker(\pi_{*p})$ for each $p \in \mathbf{M}$, then $\mathcal{V}(\mathbf{M})_p$ coincides with the tangent space of $\pi^{-1}(b)$ at p, where $\pi(p) = b$. $\mathcal{V}(\mathbf{M})_p$ is called the vertical subspace at p. There is a corresponding smooth distribution $\mathcal{V}(\mathbf{M})$ to the foliation of $\mathbf{M}$ determined by the fibers of π, called the vertical distribution. Note that the vertical vector fields are nothing but the sections of $\mathcal{V}(\mathbf{M})$. The distribution $\mathcal{H}(\mathbf{M}) = \mathcal{V}(\mathbf{M})^{\perp}$ is the complementary distribution of $\mathcal{V}(\mathbf{M})$ determined by the Riemannian metric g_m and is called the horizontal distribution. Then, for p in $\mathbf{M}$ we have the orthogonal decomposition $T_p(\mathbf{M}) = \mathcal{V}(\mathbf{M})_p \oplus \mathcal{H}(\mathbf{M})_p$, where $\mathcal{H}(\mathbf{M})_p$ is called the horizontal subspace at p. For a submersion $\pi : \mathbf{M} \longrightarrow \mathbf{B}$, let $\mathcal{H}$ and $\mathcal{V}$ denote the projection of the tangent space of $\mathbf{M}$ onto horizontal and vertical subspaces, respectively.

Definition 3.2. Let $(\mathbf{M}, g_m)$, $(\mathbf{B}, g_b)$ be semi-Riemannian manifolds of dimensions n, m, respectively $(n > m)$. A submersion $\pi : \mathbf{M} \longrightarrow \mathbf{B}$ is called a semi-Riemannian submersion if all the fibers are semi-Riemannian submanifolds of $\mathbf{M}$ and π_* preserves the length of horizontal vectors.

Note. For a semi-Riemannian submersion $\pi : (\mathbf{M}, g_m) \longrightarrow (\mathbf{B}, g_b)$, the map π_{*p} determines a linear isomorphism between $\mathcal{H}(\mathbf{M})_p$ and $T_{\pi(p)}\mathbf{B}$. Then,

$$(g_m)_p(X, Y) = (g_b)_{\pi(p)}(\pi_* X, \pi_* Y), \quad X, Y \text{for} \in \mathcal{H}(\mathbf{M})_p \quad \text{and} \quad p \in \mathbf{M}. \tag{3.1}$$

Definition 3.3. A vector field Y on $\mathbf{M}$ is said to be projectable if there exists a vector field Y_* on $\mathbf{B}$ such that $\pi_*(Y_p) = Y_{*\pi(p)}$ for each $p \in \mathbf{M}$, that is, Y and Y_* are π related. A vector field X on $\mathbf{M}$ is said to be basic if it is projectable and horizontal. Every vector field X on $\mathbf{B}$ has a unique smooth horizontal lift, denoted by $\tilde{X}$, to $\mathbf{M}$.

Proposition 3.1. *[25] Let $\pi : (\mathbf{M}, g_m) \longrightarrow (\mathbf{B}, g_b)$ be a semi-Riemannian submersion. Let X and Y in $\mathcal{X}(\mathbf{M})$ be π related to X' and Y' in $\mathcal{X}(\mathbf{B})$, respectively. Then,*

(1) $g_m(X, Y) = g_b(X', Y') \circ \pi$.

(2) $\mathcal{H}[X, Y]$ *is a basic vector field and is π related to* $[X', Y']$.

(3) $\mathcal{H}(\overset{g_m}{\nabla}_X Y)$ *is a basic vector field and is π related to* $\overset{g_b}{\nabla}_{X'} Y'$.

(4) $[X, V]$ *is vertical, for any vertical vector field V.*

$\overset{g_m}{\nabla}$ *and* $\overset{g_b}{\nabla}$ *are the Levi-Civita connections of* $\mathbf{M}$ *and* $\mathbf{B}$*, respectively.*

Proof. Since $X' = \pi_* X$ and $Y' = \pi_* Y$, property (1) follows from (3.1). Moreover, $[X, Y]$ is π related to $[X', Y']$ and (2) follows. Since $\overset{g_m}{\nabla}$ is the Levi-Civita connection on $\mathbf{M}$,

$$\begin{aligned} 2g_m(\overset{g_m}{\nabla}_X Y, Z) &= X g_m(Y, Z) + Y g_m(X, Z) - Z g_m(X, Y) \\ &\quad + g_m([X, Y], Z) + g_m([Z, X], Y) - g_m([Y, Z], X), \end{aligned} \tag{3.2}$$

for $X, Y, Z \in \mathcal{X}(\mathbf{M})$. Since X, Y, Z are the horizontal lifts of the vector fields $X^{'}, Y^{'}, Z^{'}$, we get $Xg_m(Y, Z) = X^{'}g_b(Y^{'}, Z^{'}) \circ \pi$ and $g_m([X, Y], Z) = g_b([X^{'}, Y^{'}], Z^{'}) \circ \pi$. Now, from equation (3.2)

$$g_b(\pi_*(\mathcal{H}\overset{g_m}{\nabla}_X Y), Z^{'}) \circ \pi = g_b(\overset{g_b}{\nabla}_{X^{'}} Y^{'}, Z^{'}) \circ \pi.$$

Since π is onto and $Z^{'}$ is arbitrarily chosen,

$$\pi_*(\mathcal{H}(\overset{g_m}{\nabla}_X Y)) = \overset{g_b}{\nabla}_{X^{'}} Y^{'}.$$

Hence, the property (3) follows. Finally, for any vertical vector field V, $[X, V]$ is π related to $[X^{'}, 0]$, where X is a basic vector field π related to $X^{'}$. Since $[X^{'}, 0]$ is vertical, property (4) follows. □

Definition 3.4. Let $\pi : (\mathbf{M}, g_m) \longrightarrow (\mathbf{B}, g_b)$ be a semi-Riemannian submersion. Then, the fundamental tensors T and A are defined as

$$T_E F = \mathcal{H}(\overset{g_m}{\nabla}_{\mathcal{V}E} \mathcal{V}F) + \mathcal{V}(\overset{g_m}{\nabla}_{\mathcal{V}E} \mathcal{H}F), \tag{3.3}$$

$$A_E F = \mathcal{V}(\overset{g_m}{\nabla}_{\mathcal{H}E} \mathcal{H}F) + \mathcal{H}(\overset{g_m}{\nabla}_{\mathcal{H}E} \mathcal{V}F), \tag{3.4}$$

for E, F in $\mathcal{X}(\mathbf{M})$. Here $\overset{g_m}{\nabla}$ denote the Levi-Civita connection of $(\mathbf{M}, g_m)$.

Note that these are $(1, 2)$-tensors and these tensors can be defined in a general situation, namely, it is enough that a manifold $\mathbf{M}$ has a splitting $T\mathbf{M} = \mathcal{V}(\mathbf{M}) \bigoplus \mathcal{H}(\mathbf{M})$. The following properties of T and A are often used in the subsequent calculations.

(1) T_E and A_E are, at each point, skew-symmetric linear operators on the tangent space of $\mathbf{M}$; each sends horizontal vectors to vertical, and vertical to horizontal.

(2) T is vertical and A is horizontal. That is, $T_E = T_{\mathcal{V}(E)}$ and $A_E = A_{\mathcal{H}E}$.

(3) For vertical vector fields V and W, $T_V W = T_W V$; for horizontal vector fields, X and Y, $A_X Y = \frac{1}{2}\mathcal{V}[X, Y] = -A_Y X$, [18].

Let R be the Riemannian curvature of $(\mathbf{M}, g_m)$ defined by

$$R(E, F)G = \overset{g_m}{\nabla}_{[X,Y]} G - \overset{g_m}{\nabla}_E \overset{g_m}{\nabla}_F G + \overset{g_m}{\nabla}_F \nabla_E G,$$

for $E, F, G \in \mathcal{X}(\mathbf{M})$. Let R^* denote the $(1, 3)$-tensor field on $\mathcal{X}(\mathcal{H}(\mathbf{M}))$ with values in $\mathcal{X}(\mathcal{H}(\mathbf{M}))$. That is, to any $X, Y, Z \in \mathcal{X}(\mathcal{H}(\mathbf{M}))$ and $p \in \mathbf{M}$ the tensor R^* associates the horizontal lift $R^*(X,Y,Z)_p$ of $R^{'}_{\pi(p)}(\pi_{*p}(X_p), \pi_{*p}(Y_p), \pi_{*p}(Z_p))$, where $\mathcal{X}(\mathcal{H}(\mathbf{M}))$ denote the collection of all horizontal vector fields and $R^{'}$ is the Riemannian curvature of $(\mathbf{B}, g_b)$. Then,

$$\pi_*(R^*(X,Y,Z)) = R^{'}(\pi_*X, \pi_*Y, \pi_*Z).$$

As in [50], for $E, F, G, H \in \mathcal{X}(\mathbf{M})$

$$R(E,F,G,H) \;=\; g_m(R(G,H,F),E).$$

Then, using the above notation

$$\begin{aligned} R^*(X,Y,Z,H) &= g_m(R^*(Z,H,Y),X) \\ &= R^{'}(\pi_*X, \pi_*Y, \pi_*Z, \pi_*H)\circ\pi. \end{aligned}$$

The fundamental equations are [25], [18].

$$\begin{aligned}
R(U,V,F,W) &= \hat{R}(U,V,F,W) + g_m(T_U W, T_V F) - g_m(T_V W, T_U F), && (3.5)\\
R(U,V,W,X) &= g_m((\overset{g_m}{\nabla}_V T)(U,W),X) - g_m((\overset{g_m}{\nabla}_U T)(V,W),X), && (3.6)\\
R(X,Y,Z,V) &= g_m((\overset{g_m}{\nabla}_Z A)(X,Y),V) + g_m(A_X Y, T_V Z) \\
&\quad - g_m(A_Y Z, T_V X) - g_m(A_Z X, T_V Y), && (3.7)\\
R(X,Y,Z,H) &= R^*(X,Y,Z,H) - 2g_m(A_X Y, A_Z H) \\
&\quad + g_m(A_Y Z, A_X H) - g_m(A_X Z, A_Y H), && (3.8)\\
R(X,Y,V,W) &= g_m((\overset{g_m}{\nabla}_V A)(X,Y),W) - g_m((\overset{g_m}{\nabla}_W A)(X,Y),V) \\
&\quad + g_m(A_X V, A_Y W) - g_m(A_X W, A_Y V) \\
&\quad - g_m(T_V X, T_W Y) + g_m(T_W X, T_V Y), && (3.9)\\
R(X,V,Y,W) &= g_m((\overset{g_m}{\nabla}_X T)(V,W),Y) + g_m((\overset{g_m}{\nabla}_V A)(X,Y),W) \\
&\quad + g_m(A_X V, A_Y W) - g_m(T_V X, T_W Y), && (3.10)
\end{aligned}$$

for vertical vector fields U, V, W, F and horizontal vector fields X, Y, Z, H. Here $\hat{R}$ stands for the Riemannian curvature of the fibre $(\pi^{-1}(b), \hat{g}_b)$. Note that $\hat{g}_b$ is the Riemannian metric induced by g_m on the submanifold $\pi^{-1}(b)$ of $\mathbf{M}$.

In [20], O'Neill compares the geodesics for a semi-Riemannian submersion. Let $\mathbf{M}$, $\mathbf{B}$ be Riemannian manifolds and $\pi : \mathbf{M} \to \mathbf{B}$ be a semi-Riemannian submersion. Let E be a vector field on a curve σ in $\mathbf{M}$ and the horizontal part $\mathcal{H}(E)$ and the vertical part $\mathcal{V}(E)$ of

E be denoted by H and V, respectively. Let $\pi \circ \sigma$ be a curve in B and E_* denote the vector field $\pi_*(E) = \pi_*(H)$ on the curve $\pi \circ \sigma$ in $\mathbf{B}$. $E_*^{'}$ denote the covariant derivative of E_* and is a vector field on $\pi \circ \sigma$. The horizontal lift to σ of $E_*^{'}$ is denoted by $\tilde{E}_*^{'}$. O'Neill [20] has shown that

$$\begin{aligned} \mathcal{H}(E^{'}) &= \tilde{E}_*^{'} + A_H(U) + A_X(V) + T_U(V), \quad (3.11) \\ \mathcal{V}(E^{'}) &= A_X H + T_U H + \mathcal{V}(V^{'}), \quad (3.12) \end{aligned}$$

where $X = \mathcal{H}(\sigma^{'})$ and $U = \mathcal{V}(\sigma^{'})$ and A, T are fundamental tensors.
Acceleration $\sigma^{''}$ of σ is the covariant derivative of $\sigma^{'}$. Then,

Proposition 3.2. *[20] Let σ be a curve in $\mathbf{M}$ with $X = \mathcal{H}(\sigma^{'})$ and $U = \mathcal{V}(\sigma^{'})$. Then,*

$$\begin{aligned} \mathcal{H}(\sigma^{''}) &= \tilde{\sigma}_*^{''} + 2A_X U + T_U U, \quad (3.13) \\ \mathcal{V}(\sigma^{''}) &= T_U X + \mathcal{V}(U^{'}), \quad (3.14) \end{aligned}$$

where $\sigma_*^{''}$ is the acceleration of $\pi \circ \sigma$.

Proof. From property (3) of the fundamental tensors mentioned above, $A_X X = 0$ for the horizontal vector field X. Now, by setting $H = \mathcal{H}(\sigma^{'})$ and $V = \mathcal{V}(\sigma^{'})$ in equations (3.11) and (3.12)

$$\begin{aligned} \mathcal{H}(\sigma^{''}) &= \tilde{\sigma}_*^{''} + 2A_X U + T_U U. \\ \mathcal{V}(\sigma^{''}) &= T_U X + \mathcal{V}(U^{'}). \end{aligned}$$

□

Remark 3.1. From equations (3.13) and (3.14) it can be observed that, if σ is a geodesic of $\mathbf{M}$, then $\pi \circ \sigma$ is a geodesic of $\mathbf{B}$ if and only if $2A_X U + T_U U = 0$. In particular if σ is a horizontal geodesic ($U = \mathcal{V}(\sigma^{'}) = 0$), then $\pi \circ \sigma$ is a geodesic.

Definition 3.5. Let $\pi : (\mathbf{M}, \nabla, g_m) \to (\mathbf{B}, \nabla^*, g_b)$ be a semi-Riemannian submersion and α be a smooth curve in $\mathbf{B}$. Let $\alpha^{'}$ be the tangent vector field of α and $(\tilde{\alpha^{'}})$ be its horizontal lift. Define the horizontal lift of the curve α as the integral curve σ in $\mathbf{M}$ of $(\tilde{\alpha^{'}})$.

Proposition 3.3. *[20] Let $\pi : (\mathbf{M}, g_m) \to (\mathbf{B}, g_b)$ be a semi-Riemannian submersion. Then, every horizontal lift of a geodesic of $\mathbf{B}$ is a geodesic of $\mathbf{M}$.*

Proof. Let α be a geodesic of $\mathbf{B}$ and σ be the horizontal lift of α. Then, $\pi \circ \sigma = \alpha$ and $\sigma^{'}(t) = (\alpha^{'}(t))\tilde{}$. Let $X = \mathcal{H}(\sigma^{'}(t))$ and $U = \mathcal{V}(\sigma^{'}(t))$, clearly $X = (\alpha^{'}(t))\tilde{}$ and $U = 0$. Then, from (3.13) and (3.14)

$$\begin{aligned} \mathcal{H}(\sigma^{''}) &= \alpha^{''}. \\ \mathcal{V}(\sigma^{''}) &= 0. \end{aligned}$$

Since α is a geodesic we get $\alpha^{''} = 0$. Hence σ is a geodesic of $\mathbf{M}$. $\square$

Remark 3.2. O'Neill [20] proved that for a semi-Riemannian submersion $\pi : \mathbf{M} \longrightarrow \mathbf{B}$, if γ is a geodesic of $\mathbf{M}$ that is horizontal at some point, then γ is always horizontal (that is, $\pi \circ \gamma$ is geodesic of $\mathbf{B}$). Hence, $\mathbf{B}$ is geodesically complete if $\mathbf{M}$ is geodesically complete.

3.2 Affine Submersion with Horizontal Distribution

In this section, we give the definition and basic results of an affine submersion with horizontal distribution for Riemannian manifolds. Also, discuss the theorem by Abe and Hasegawa [21] on geodesics comparison for an affine submersion with horizontal distribution.

Suppose $\mathbf{M}$ and $\mathbf{B}$ are smooth manifolds of dimensions m and n $(m > n)$, respectively.

Definition 3.6. A submersion $\pi : \mathbf{M} \longrightarrow \mathbf{B}$ is called a submersion with horizontal distribution if there is a smooth distribution $p \longrightarrow \mathcal{H}(\mathbf{M})_p$ such that

$$T_p\mathbf{M} = \mathcal{V}(\mathbf{M})_p \bigoplus \mathcal{H}(\mathbf{M})_p, \tag{3.15}$$

where $\mathcal{V}(\mathbf{M})_p = ker(\pi_{*p})$ for each $p \in \mathbf{M}$.

Note. Projectable and basic vector fields are defined as in the case of semi-Riemann submersion. A vector field X on $\mathbf{B}$ has a unique smooth horizontal lift, denoted by $\tilde{X}$, to $\mathbf{M}$.

Definition 3.7. Let ∇ and ∇^* be affine connections on $\mathbf{M}$ and $\mathbf{B}$, respectively.
$\pi : (\mathbf{M}, \nabla) \longrightarrow (\mathbf{B}, \nabla^*)$ is said to be an affine submersion with horizontal distribution if $\pi : \mathbf{M} \longrightarrow \mathbf{B}$ is a submersion with horizontal distribution and satisfies $\mathcal{H}(\nabla_{\tilde{X}}\tilde{Y}) = (\nabla^*_X Y)\tilde{}$, for $X, Y \in \mathcal{X}(\mathbf{B})$.

Note. If $\pi : (\mathbf{M}, g_m) \longrightarrow (\mathbf{B}, g_b)$ is a semi-Riemannian submersion and $\overset{g_m}{\nabla}$ and $\overset{g_b}{\nabla}$ are Levi-Civita connections on $\mathbf{M}$ and $\mathbf{B}$, respectively, then $\pi : (\mathbf{M}, \overset{g_m}{\nabla}) \longrightarrow (\mathbf{B}, \overset{g_m}{\nabla})$ is obviously an affine submersion with horizontal distribution $\mathcal{H}(\mathbf{M}) = \mathcal{V}(\mathbf{M})^{\perp}$.

Theorem 3.1. *[21] Assume that $\pi : \mathbf{M} \longrightarrow \mathbf{B}$ is a submersion with horizontal distribution and ∇ is an affine connection on $\mathbf{M}$. If $\mathcal{H}(\nabla_{\tilde{X}}\tilde{Y})$ is projectable for all vector fields X and Y on $\mathbf{B}$, then there exists a unique affine connection ∇^* on $\mathbf{B}$ such that $\pi : (\mathbf{M}, \nabla) \longrightarrow (\mathbf{B}, \nabla^*)$ is an affine submersion with horizontal distribution.*

Proof. Setting $\nabla^*_X Y = \pi_*(\nabla_{\tilde{X}}\tilde{Y})$, we show that ∇^* is an affine connection on $\mathbf{B}$. Since $(\tilde{fX}) = (f \circ \pi)\tilde{X}$,

$$
\begin{aligned}
\nabla^*_X(fY) &= \pi_*(\nabla_{\tilde{X}}(f \circ \pi)\tilde{Y}) = \pi_*((\tilde{X}(f \circ \pi))\tilde{Y} + (f \circ \pi)\nabla_{\tilde{X}}\tilde{Y}) \\
&= (Xf)Y + f\nabla^*_{\tilde{X}}\tilde{Y}.
\end{aligned}
$$

The other conditions for the affine connection can be proved similarly. The uniqueness is clear from the definition. □

Note. A connection $\mathcal{V}\nabla\mathcal{V}$ on the subbundle $\mathcal{V}(\mathbf{M})$ is defined by $(\mathcal{V}\nabla\mathcal{V})_E V = \mathcal{V}(\nabla_E V)$, for any vertical vector field V and any vector field E on $\mathbf{M}$. For each $b \in \mathbf{B}$, $\mathcal{V}\nabla\mathcal{V}$ induces a unique connection $\hat{\nabla}^b$ on the fiber $\pi^{-1}(b)$. Hereafter we often omit the superscript b. In [21], Abe and Hasegawa observed that if ∇ is torsion-free, then $\hat{\nabla}^b$ and ∇^* are also torsion-free.

As we mentioned in the previous section, to define fundamental tensors, we only need to split the tangent bundle $T\mathbf{M} = \mathcal{V}(\mathbf{M}) \bigoplus \mathcal{H}(\mathbf{M})$. The fundamental tensors for an affine submersion with horizontal distribution are defined as follows.

Definition 3.8. Let $\pi : (\mathbf{M}, \nabla, g_m) \longrightarrow (\mathbf{B}, \nabla^*, g_b)$ be an affine submersion with horizontal distribution $\mathcal{H}(\mathbf{M})$. Then, the fundamental tensors T and A are defined as

$$
\begin{aligned}
T_E F &= \mathcal{H}(\nabla_{\mathcal{V}E}\mathcal{V}F) + \mathcal{V}(\nabla_{\mathcal{V}E}\mathcal{H}F), \\
A_E F &= \mathcal{V}(\nabla_{\mathcal{H}E}\mathcal{H}F) + \mathcal{H}(\nabla_{\mathcal{H}E}\mathcal{V}F),
\end{aligned}
$$

for E and F in $\mathcal{X}(\mathbf{M})$. Also, the fundamental tensors corresponding to the dual connection $\overline{\nabla}$ of ∇ are denoted by $\overline{T}$ and $\overline{A}$.

These are $(1, 2)$-tensors. Also, T_E and A_E reverses the horizontal and vertical subspaces and $T_E = T_{\mathcal{V}E}$, $A_E = A_{\mathcal{H}E}$.

The inclusion map $(\pi^{-1}(b), \hat{\nabla}^b) \longrightarrow (\mathbf{M}, \nabla)$ is an affine immersion [2]. The following equations are corresponding to the Gauss and the Weingarten formulae. Let X and Y be horizontal vector fields and V and W be vertical vector fields on $\mathbf{M}$. Then,

$$\begin{aligned}
\nabla_V W &= T_V W + \hat{\nabla}_V W. && (3.16)\\
\nabla_V X &= \mathcal{H}(\nabla_V X) + T_V X. && (3.17)\\
\nabla_X V &= \mathcal{V}(\nabla_X V) + A_X V. && (3.18)\\
\nabla_X Y &= \mathcal{H}(\nabla_X Y) + A_X Y. && (3.19)
\end{aligned}$$

Let R be the curvature tensor of $(\mathbf{M}, \nabla)$. Denote the curvature tensor of ∇^* (respectively $\hat{\nabla}$) by R^* (respectively $\hat{R}$). Define $(1,3)$-tensors R^{P_1,P_2,P_3} for affine submersion with horizontal distribution by

$$\begin{aligned}
R^{P_1,P_2,P_3}(E,F)G &= P_3\nabla_{[P_1E,P_2F]}P_3G - P_3\nabla_{P_1E}(P_3\nabla_{P_2F}P_3G)\\
&\quad +P_3\nabla_{P_2F}(P_3\nabla_{P_1E}P_3G),
\end{aligned}$$

where $P_i = \mathcal{H}$ or $\mathcal{V}$ for $i = 1,2,3$ and E, F, G are vector fields on $\mathbf{M}$. The fundamental equations for affine submersion with horizontal distribution are [21].

$$\begin{aligned}
\mathcal{V}R(U,V)W &= R^{\mathcal{V}\mathcal{V}\mathcal{V}}(U,V)W + T_V T_U W - T_U T_V W,\\
\mathcal{H}R(U,V)W &= \mathcal{H}(\nabla_V T)_U W - \mathcal{H}(\nabla_U T)_V W - T_{Tor(\nabla)(U,V)}W,\\
\mathcal{V}R(U,V)X &= \mathcal{H}(\nabla_V T)_U X - \mathcal{V}(\nabla_U T)_V X - T_{Tor(\nabla)(U,V)}X,\\
\mathcal{H}R(U,V)X &= R^{\mathcal{V}\mathcal{V}\mathcal{H}}(U,V)X + T_V T_U X - T_U T_V X,\\
\mathcal{V}R(U,X)V &= R^{\mathcal{V}\mathcal{H}\mathcal{V}}(U,X)V - T_U A_X V - A_X T_U V,\\
\mathcal{H}R(U,X)V &= \mathcal{H}(\nabla_X T)_U V - \mathcal{H}(\nabla_U A)_X V - A_{A_X U}V + T_{T_U X}V\\
&\quad -T_{Tor(\nabla)(U,X)}V - A_{Tor(\nabla)(U,X)}V,\\
\mathcal{V}R(U,X)Y &= \mathcal{V}(\nabla_X T)_U Y - \mathcal{V}(\nabla_U A)_X Y - A_{A_X U}Y + T_{T_U X}Y\\
&\quad -T_{Tor(\nabla)(U,X)}Y - A_{Tor(\nabla)(U,X)}Y,\\
\mathcal{H}R(U,X)Y &= R^{\mathcal{V}\mathcal{H}\mathcal{H}}(U,X)Y - T_V A_X Y + A_X T_U Y,\\
\mathcal{V}R(X,Y)U &= R^{\mathcal{H}\mathcal{H}\mathcal{V}}(X,Y)U + A_Y A_X U - A_X A_Y U,
\end{aligned}$$

$$
\begin{aligned}
\mathcal{H}R(X,Y)U &= \mathcal{H}(\nabla_Y A)_X U - \mathcal{H}(\nabla_X A)_Y U + T_{A_X Y}U - T_{A_Y X}U \\
&\quad - T_{Tor(\nabla)(X,Y)}U - A_{Tor(\nabla)(X,Y)}U, \\
\mathcal{V}R(X,Y)Z &= \mathcal{V}(\nabla_Y A)_X Z - \mathcal{V}(\nabla_X A)_Y Z + T_{A_X Y}Z - T_{A_Y X}Z \\
&\quad - T_{Tor(\nabla)(X,Y)}Z - A_{Tor(\nabla)(X,Y)}Z, \\
\mathcal{H}R(X,Y)Z &= R^{\mathcal{H}\mathcal{H}\mathcal{H}}(X,Y)Z + A_Y A_X Z - A_X A_Y Z,
\end{aligned}
$$

where X, Y, Z are horizontal vector fields and U, V, W are vertical vector fields.

In the previous section, the comparison of geodesics for semi-Riemannian submersion is discussed. In [21], Abe and Hasegawa compared the geodesics of $\mathbf{M}$ and $\mathbf{B}$ for affine submersion with horizontal distribution. Let $\pi : (\mathbf{M}, \nabla) \rightarrow (\mathbf{B}, \nabla^*)$ be an affine submersion with horizontal distribution. Let E be a vector field on a curve σ in $\mathbf{M}$ and the horizontal part $\mathcal{H}(E)$ and the vertical part $\mathcal{V}(E)$ of E be denoted by H and V, respectively. $\pi \circ \sigma$ is a curve in B and E_* denote the vector field $\pi_*(E) = \pi_*(H)$ on the curve $\pi \circ \sigma$ in $\mathbf{B}$. $E_*^{'}$ denote the covariant derivative of E_* and is a vector field on $\pi \circ \sigma$. The horizontal lift to σ of $E_*^{'}$ is denoted by $\tilde{E_*^{'}}$. In [21], Abe and Hasegawa have shown that

$$
\begin{aligned}
\mathcal{H}(E^{'}) &= E_*^{'} + \mathcal{H}Tor(\nabla)(U, H) + A_H U + A_X V + T_U V, && (3.20) \\
\mathcal{V}(E^{'}) &= A_X H + T_U H + \mathcal{V}(V^{'}), && (3.21)
\end{aligned}
$$

where $X = \mathcal{H}(\sigma^{'})$ and $U = \mathcal{V}(\sigma^{'})$. Here $Tor(\nabla)$ denote the torsion of ∇.

Proposition 3.4. *[21] Let σ be a curve in $\mathbf{M}$ with $X = \mathcal{H}(\sigma^{'})$ and $U = \mathcal{V}(\sigma^{'})$. Then,*

$$
\begin{aligned}
\mathcal{H}(\sigma^{''}) &= \sigma_*^{''} + \mathcal{H}Tor(\nabla)(U, X) + 2A_X U + T_U U, \\
\mathcal{V}(\sigma^{''}) &= A_X X + T_U X + \mathcal{V}(U^{'}),
\end{aligned}
$$

where $\sigma_*^{''}$ denotes the covariant derivative of $(\pi \circ \sigma)^{'}$.

Proof. By setting $H = \mathcal{H}(\sigma^{'})$ and $V = \mathcal{V}(\sigma^{'})$ in equations (3.20) and (3.21)

$$
\begin{aligned}
\mathcal{H}(\sigma^{''}) &= \sigma_*^{''} + \mathcal{H}Tor(\nabla)(U, X) + 2A_X U + T_U U \\
\mathcal{V}(\sigma^{''}) &= A_X X + T_U X + \mathcal{V}(U^{'}),
\end{aligned}
$$

□

Remark 3.3. From above proposition it can be seen that if σ is a geodesic, then $\pi \circ \sigma$ is a geodesic of $\mathbf{B}$ if and only if $\mathcal{H}Tor(\nabla)(U, X) + 2A_X U + T_U U = 0$. In particular, if σ is a horizontal geodesic $(ie, U = \mathcal{V}(\sigma^{'}) = 0)$, then $\pi \circ \sigma$ is a geodesic.

Corollary 3.1. *[21] Let $\pi : (\mathbf{M}, \nabla) \longrightarrow (\mathbf{B}, \nabla^*)$ be an affine submersion with horizontal distribution $\mathcal{H}(\mathbf{M})$ such that $A_X X = 0$ for all horizontal vector X. Then, every horizontal lift of a geodesic of $\mathbf{B}$ is a geodesic of $\mathbf{M}$.*

Proof. Proof follows from the proposition (3.4). □

Remark 3.4. In [21], Abe and Hasegawa proved that for an affine submersion $\pi : (\mathbf{M}, \nabla) \rightarrow (\mathbf{B}, \nabla^*)$ with horizontal distribution $\mathcal{H}(\mathbf{M})$ such that $A_Z Z = 0$ for all horizontal vectors Z, ∇^* is geodesically complete if ∇ is geodesically complete.

3.3 Conformal Submersion with Horizontal Distribution

In this section, we introduce the concept of a conformal submersion with horizontal distribution for Riemannian manifolds, which is a generalization of the affine submersion with horizontal distribution. Then, a necessary condition for the existence of such a map is given. A necessary and sufficient condition is obtained for $\pi \circ \sigma$ to be a geodesic of $\mathbf{B}$ when σ is a geodesic of $\mathbf{M}$ for a conformal submersion with horizontal distribution. Then, proved a necessary and sufficient condition for the horizontal lift of a geodesic to be geodesic. Also, we give a necessary condition for the connection on $\mathbf{B}$ to be complete when the coneection on $\mathbf{M}$ is complete for a conformal submersion with horizontal distribution $\pi : \mathbf{M} \longrightarrow \mathbf{B}$.

Definition 3.9. Let $(\mathbf{M}, g_m)$ and $(\mathbf{B}, g_b)$ be Riemannian manifolds. A submersion $\pi : (\mathbf{M}, g_m) \longrightarrow (\mathbf{B}, g_b)$ is called a conformal submersion if there exists a $\phi \in C^\infty(\mathbf{M})$ such that $g_m(X, Y) = e^{2\phi} g_b(\pi_* X, \pi_* Y)$, for horizontal vector fields $X, Y \in \mathcal{X}(\mathbf{M})$.

For $\pi : (\mathbf{M}, \nabla) \longrightarrow (\mathbf{B}, \nabla^*)$ an affine submersion with horizontal distribution, $\pi_*(\nabla_{\tilde{X}} \tilde{Y}) = \nabla^*_X Y$, for $X, Y \in \mathcal{X}(\mathbf{B})$. In the case of conformal submersion we prove the following theorem, which is the motivation for us to generalize the concept of an affine submersion with horizontal distribution.

Theorem 3.2. *Let $\pi : (\mathbf{M}, g_m) \longrightarrow (\mathbf{B}, g_b)$ be a conformal submersion. If $\overset{g_m}{\nabla}$ on $\mathbf{M}$ and $\overset{g_b}{\nabla}$ on $\mathbf{B}$ are the Levi-Civita connections, then*

$$\begin{aligned} g_b(\pi_*(\overset{g_m}{\nabla}_{\tilde{X}} \tilde{Y}), Z) &= g_b(\overset{g_b}{\nabla}_X Y, Z) - d\phi(\tilde{Z}) g_b(X, Y) \\ &\quad + \{d\phi(\tilde{X}) g_b(Y, Z) + d\phi(\tilde{Y}) g_b(Z, X)\}, \end{aligned}$$

where $X, Y, Z \in \mathcal{X}(\mathbf{B})$ and $\tilde{X}, \tilde{Y}, \tilde{Z}$ denote the unique horizontal lifts on $\mathbf{M}$.

Proof. We have the Koszul formula for the Levi-Civita connection,

$$\begin{aligned}2g_m(\overset{g_m}{\nabla}_{\tilde{X}}\tilde{Y},\tilde{Z}) &= \tilde{X}g_m(\tilde{Y},\tilde{Z})+\tilde{Y}g_m(\tilde{Z},\tilde{X})-\tilde{Z}g_m(\tilde{X},\tilde{Y})\\ &\quad -g_m(\tilde{X},[\tilde{Y},\tilde{Z}])+g_m(\tilde{Y},[\tilde{Z},\tilde{X}])+g_m(\tilde{Z},[\tilde{X},\tilde{Y}]).\end{aligned} \tag{3.22}$$

Now, consider

$$\begin{aligned}\tilde{X}g_m(\tilde{Y},\tilde{Z}) &= \tilde{X}(e^{2\phi}g_b(Y,Z))\\ &= \tilde{X}(e^{2\phi})g_b(Y,Z)+e^{2\phi}\tilde{X}(g_b(Y,Z))\\ &= 2e^{2\phi}d\phi(\tilde{X})g_b(Y,Z)+e^{2\phi}Xg_b(Y,Z).\end{aligned}$$

Similarly,

$$\begin{aligned}\tilde{Y}g_m(\tilde{X},\tilde{Z}) &= 2e^{2\phi}d\phi(\tilde{Y})g_b(X,Z)+e^{2\phi}Yg_b(X,Z).\\ \tilde{Z}g_m(\tilde{X},\tilde{Y}) &= 2e^{2\phi}d\phi(\tilde{Z})g_b(X,Y)+e^{2\phi}Zg_b(X,Y).\end{aligned}$$

Also, $g_m(\tilde{X},[\tilde{Y},\tilde{Z}])=e^{2\phi}g_b(X,[Y,Z])$, $g_m(\tilde{Y},[\tilde{Z},\tilde{X}])=e^{2\phi}g_b(Y,[Z,X])$ and $g_m(\tilde{Z},[\tilde{X},\tilde{Y}])=e^{2\phi}g_b(Z,[X,Y])$. Then, from the equation (3.22) and the above equations

$$\begin{aligned}2g_m(\overset{g_m}{\nabla}_{\tilde{X}}\tilde{Y},\tilde{Z}) &= 2d\phi(\tilde{X})e^{2\phi}g_b(Y,Z)+2d\phi(\tilde{Y})e^{2\phi}g_b(X,Z)\\ &\quad -2d\phi(\tilde{Z})e^{2\phi}g_b(X,Y)+2e^{2\phi}g_b(\overset{g_b}{\nabla}_XY,Z).\end{aligned}$$

This implies

$$\begin{aligned}g_b(\pi_*(\overset{g_m}{\nabla}_{\tilde{X}}\tilde{Y}),Z) &= g_b(\overset{g_b}{\nabla}_XY,Z)-d\phi(\tilde{Z})g_b(X,Y)\\ &\quad +\{d\phi(\tilde{X})g_b(Y,Z)+d\phi(\tilde{Y})g_b(Z,X)\}.\end{aligned}$$

□

Now, we generalize the concept of an affine submersion with horizontal distribution as follows:

Definition 3.10. Let $\pi:(\mathbf{M},g_m)\longrightarrow(\mathbf{B},g_b)$ be a conformal submersion and let ∇ and ∇^* be affine connections on $\mathbf{M}$ and $\mathbf{B}$, respectively. Then, $\pi:(\mathbf{M},\nabla)\longrightarrow(\mathbf{B},\nabla^*)$ is said to be a conformal submersion with horizontal distribution $\mathcal{H}(\mathbf{M}) = \mathcal{V}(\mathbf{M})^{\perp}$ if

$$\begin{aligned} g_b(\pi_*(\nabla_{\tilde{X}}\tilde{Y}), Z) &= g_b(\nabla^*_X Y, Z) - d\phi(\tilde{Z})g_b(X,Y) \\ &\quad +\{d\phi(\tilde{X})g_b(Y,Z) + d\phi(\tilde{Y})g_b(Z,X)\}, \end{aligned} \tag{3.23}$$

for some $\phi \in C^{\infty}(\mathbf{M})$ and for $X, Y, Z \in \mathcal{X}(\mathbf{B})$.

Note. If ϕ is constant, it turns out to be an affine submersion with horizontal distribution. Also, the equation (3.23) can be written as

$$\mathcal{H}(\nabla_{\tilde{X}}\tilde{Y}) = (\widetilde{\nabla^*_X Y}) + \tilde{X}(\phi)\tilde{Y} + \tilde{Y}(\phi)\tilde{X} - \mathcal{H}(grad_{\pi}\phi)g_m(\tilde{X},\tilde{Y}).$$

Example 3.1. Let $H^n = \{(x_1, ..., x_n) \in \mathbf{R}^n : x_n > 0\}$ and $\tilde{g} = \frac{1}{x_n^2}g$ be a Riemannian metric on H^n, where g is the Euclidean metric on $\mathbf{R}^n$. Let $\pi : H^n \longrightarrow \mathbf{R}^{n-1}$ be defined by $\pi(x_1, ..., x_n) = (x_1, ..., x_{n-1})$. Let $\phi : H^n \longrightarrow \mathbf{R}$ be defined by $\phi(x_1, ..., x_n) = \log(\frac{1}{x_n^2})$. Then, we have

$$\tilde{g}\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = e^{\phi} g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right).$$

Hence, $\pi : (H^n, \tilde{g}) \longrightarrow (\mathbf{R}^{n-1}, g)$ is a conformal submersion. Then, by theorem (3.2), $\pi : (H^n, \overset{\tilde{g}}{\nabla}) \longrightarrow (\mathbf{R}^{n-1}, \overset{g}{\nabla})$ is a conformal submersion with horizontal distribution, where $\overset{\tilde{g}}{\nabla}$ and $\overset{g}{\nabla}$ are Levi-Civita connections on H^n and $\mathbf{R}^{n-1}$, respectively.

Now, a necessary condition is obtained for the existence of a conformal submersion with horizontal distribution.

Theorem 3.3. *Let $\pi : (\mathbf{M}, g_m) \longrightarrow (\mathbf{B}, g_b)$ be a conformal submersion and ∇ be an affine connection on $\mathbf{M}$. Assume that $\pi : \mathbf{M} \longrightarrow \mathbf{B}$ is a submersion with horizontal distribution. If $\mathcal{H}(\nabla_{\tilde{X}}\tilde{Y})$ is projectable for X, Y in $\mathcal{X}(\mathbf{B})$, then there exists a unique connection ∇^* on $\mathbf{B}$ such that $\pi : (\mathbf{M}, \nabla) \longrightarrow (\mathbf{B}, \nabla^*)$ is a conformal submersion with horizontal distribution.*

Proof. Setting $\nabla^*_X Y = \pi_*(\nabla_{\tilde{X}}\tilde{Y}) - \tilde{X}(\phi)Y - \tilde{Y}(\phi)X + e^{2\phi}\pi_*(grad_{\pi}\phi)g_b(X,Y)$, we show that ∇^* is an affine connection on $\mathbf{B}$. Since $(\widetilde{fX}) = (f \circ \pi)\tilde{X}$, we have

$$\begin{aligned} \nabla^*_X(fY) &= \pi_*(\nabla_{\tilde{X}}(f \circ \pi)\tilde{Y}) - \tilde{X}(\phi)fY - (\widetilde{fY})(\phi)X \\ &\quad +e^{2\phi}\pi_*(grad_{\pi}\phi)g_b(X, fY). \end{aligned} \tag{3.24}$$

Now, consider

$$\begin{aligned}\pi_*(\nabla_{\tilde{X}}(f\circ\pi)\tilde{Y}) &= \pi_*((\tilde{X}(f\circ\pi))\tilde{Y}+(f\circ\pi)\nabla_{\tilde{X}}\tilde{Y}) \\ &= X(f)Y+f\nabla^*_XY+\tilde{X}(\phi)fY+f\tilde{Y}(\phi)X \\ &\quad -e^{2\phi}\pi_*(grad_\pi\phi)fg_b(X,Y). \qquad (3.25)\end{aligned}$$

From (3.24) and (3.25)

$$\nabla^*_X(fY) = X(f)Y+f\nabla^*_XY.$$

The other condition for affine connection can be proved similarly. The uniqueness is clear from the definition. □

Note. From the definition of conformal submersion with horizontal distribution we get if ∇ is torsion-free, then ∇^* and $\hat{\nabla}$ are also torsion-free.

Fundamental tensors T and A for conformal submersion with horizontal distribution $\pi:(\mathbf{M},\nabla)\longrightarrow(\mathbf{B},\nabla^*)$ are defined for E and F in $\mathcal{X}(\mathbf{M})$ by

$$T_EF=\mathcal{H}\nabla_{\mathcal{V}E}(\mathcal{V}F)+\mathcal{V}\nabla_{\mathcal{V}E}(\mathcal{H}F)$$

and

$$A_EF=\mathcal{V}\nabla_{\mathcal{H}E}(\mathcal{H}F)+\mathcal{H}\nabla_{\mathcal{H}E}(\mathcal{V}F).$$

Note that these are $(1,2)$-tensors.

Let R denote the curvature tensor of $(\mathbf{M},\nabla)$, denote the curvature tensor of ∇^* (respectively $\hat{\nabla}$) by R^* (respectively $\hat{R}$). Define $(1,3)$-tensors R^{P_1,P_2,P_3} for the conformal submersion with horizontal distribution by

$$\begin{aligned}R^{P_1,P_2,P_3}(E,F)G &= P_3\nabla_{[P_1E,P_2F]}P_3G-P_3\nabla_{P_1E}(P_3\nabla_{P_2F}P_3G) \\ &\quad +P_3\nabla_{P_2F}(P_3\nabla_{P_1E}P_3G),\end{aligned}$$

where $P_i=\mathcal{H}$ or $\mathcal{V}$ $(i=1,2,3)$ and E,F,G are vector fields on $\mathbf{M}$. Then, the following fundamental equations for the conformal submersion with horizontal distribution are obtained.

Theorem 3.4. *Let X,Y,Z be horizontal vector fields and U,V,W be vertical vector fields.*

Then,

$$
\begin{aligned}
\mathcal{V}R(U,V)W &= R^{\mathcal{VVV}}(U,V)W + T_V T_U W - T_U T_V W.\\
\mathcal{H}R(U,V)W &= \mathcal{H}(\nabla_V T)_U W - \mathcal{H}(\nabla_U T)_V W - T_{Tor(\nabla)(U,V)}W.\\
\mathcal{V}R(U,V)X &= \mathcal{H}(\nabla_V T)_U X - \mathcal{V}(\nabla_U T)_V X - T_{Tor(\nabla)(U,V)}X.\\
\mathcal{H}R(U,V)X &= R^{\mathcal{VVH}}(U,V)X + T_V T_U X - T_U T_V X.\\
\mathcal{V}R(U,X)V &= R^{\mathcal{VHV}}(U,X)V - T_U A_X V - A_X T_U V.\\
\mathcal{H}R(U,X)V &= \mathcal{H}(\nabla_X T)_U V - \mathcal{H}(\nabla_U A)_X V - A_{A_X U}V + T_{T_U X}V\\
&\quad - T_{Tor(\nabla)(U,X)}V - A_{Tor(\nabla)(U,X)}V.\\
\mathcal{V}R(U,X)Y &= \mathcal{V}(\nabla_X T)_U Y - \mathcal{V}(\nabla_U A)_X Y - A_{A_X U}Y + T_{T_U X}Y\\
&\quad - T_{Tor(\nabla)(U,X)}Y - A_{Tor(\nabla)(U,X)}Y.\\
\mathcal{H}R(U,X)Y &= R^{\mathcal{VHH}}(U,X)Y - T_V A_X Y + A_X T_U Y.\\
\mathcal{V}R(X,Y)U &= R^{\mathcal{HHV}}(X,Y)U + A_Y A_X U - A_X A_Y U.\\
\mathcal{H}R(X,Y)U &= \mathcal{H}(\nabla_Y A)_X U - \mathcal{H}(\nabla_X A)_Y U + T_{A_X Y}U - T_{A_Y X}U\\
&\quad - T_{Tor(\nabla)(X,Y)}U - A_{Tor(\nabla)(X,Y)}U.\\
\mathcal{V}R(X,Y)Z &= \mathcal{V}(\nabla_Y A)_X Z - \mathcal{V}(\nabla_X A)_Y Z + T_{A_X Y}Z - T_{A_Y X}Z\\
&\quad - T_{Tor(\nabla)(X,Y)}Z - A_{Tor(\nabla)(X,Y)}Z.\\
\mathcal{H}R(X,Y)Z &= R^{\mathcal{HHH}}(X,Y)Z + A_Y A_X Z - A_X A_Y Z.
\end{aligned}
$$

3.3.1 Geodesics

In this subsection, for a conformal submersion with horizontal distribution we prove a necessary and sufficient condition for $\pi\circ\sigma$ to be a geodesic of $\mathbf{B}$ when σ is a geodesic of $\mathbf{M}$. Then, obtained a necessary and sufficient condition for the horizontal lift of a geodesic to be geodesic. Also, we give a necessary condition for the connection on $\mathbf{B}$ to be complete when the connection on $\mathbf{M}$ is complete for a conformal submersion with horizontal distribution $\pi : \mathbf{M} \longrightarrow \mathbf{B}$.

Let $\mathbf{M}$, $\mathbf{B}$ be Riemannian manifolds and $\pi : \mathbf{M} \to \mathbf{B}$ be a submersion. Let E be a vector field on a curve σ in $\mathbf{M}$ and the horizontal part $\mathcal{H}(E)$ and the vertical part $\mathcal{V}(E)$ of E be denoted by H and V, respectively. $\pi\circ\sigma$ is a curve in $\mathbf{B}$ and E_* denote the vector field $\pi_*(E) = \pi_*(H)$ on the curve $\pi\circ\sigma$ in $\mathbf{B}$. $E_*^{'}$ denote the covariant derivative of E_* and is a vector field on $\pi\circ\sigma$. The horizontal lift to σ of $E_*^{'}$ is denoted by $\tilde{E}_*^{'}$.

Let $\pi : (\mathbf{M}, \nabla, g_m) \to (\mathbf{B}, \nabla^*, g_b)$ a conformal submersion with horizontal distribution

$\mathcal{H}(\mathbf{M})$. Throughout this section we assume ∇ is torsion-free. A curve σ is a geodesic if and only if $\mathcal{H}(\sigma^{''}) = 0$ and $\mathcal{V}(\sigma^{''}) = 0$, where $\sigma^{''}$ is the covariant derivative of $\sigma^{'}$. So, first we obtain the equations for $\mathcal{H}(E^{'})$ and $\mathcal{V}(E^{'})$ for a vector field E on a curve σ in $\mathbf{M}$ for a conformal submersion with horizontal distribution.

Theorem 3.5. *Let $\pi : (\mathbf{M}, \nabla, g_m) \to (\mathbf{B}, \nabla^*, g_b)$ be a conformal submersion with horizontal distribution and let $E = H + V$ be a vector field on a curve σ in $\mathbf{M}$. Then,*

$$\begin{aligned}
\pi_*(\mathcal{H}(E^{'})) &= E^{'}_* + \pi_*(A_X U + A_X V + T_U V) - e^{2\phi}\pi_*(grad_\pi \phi) g_b(\pi_* X, \pi_* H) \\
&\quad + X(\phi)\pi_* H + H(\phi)\pi_* X, \\
\mathcal{V}(E^{'}) &= A_X H + T_U H + \mathcal{V}(V^{'}),
\end{aligned}$$

where $X = \mathcal{H}(\sigma^{'})$ and $U = \mathcal{V}(\sigma^{'})$.

Proof. Consider a neighborhood of an arbitrary point $\sigma(t)$ of the curve σ in $\mathbf{M}$. By choosing the base fields $W_1, ..., W_n$, where $n = dim\mathbf{B}$, near $\pi(\sigma(t))$ on $\mathbf{B}$ and an appropriate vertical base field near $\sigma(t)$, we have

$$(E^{'}_*)_t = \sum_i r^{i'}(t)(W_i)_{\pi(\sigma(t))} + \sum_{i,k} r^i(t) s^k(t) (\nabla^*_{W_k} W_i)_{\pi(\sigma(t))}, \tag{3.26}$$

$$\begin{aligned}
\pi_*(\mathcal{H}(E^{'})_t) &= \sum_i r^{i'}(t)(W_i)_{\pi(\sigma(t))} + \sum_{i,k} r^i(t) s^k(t) \pi_*(\mathcal{H}(\nabla_{\tilde{W}_k} \tilde{W}_i))_{\pi(\sigma(t))} \\
&\quad + \pi_*((A_H U) + (A_X V) + (T_U V))_{\pi(\sigma(t))},
\end{aligned} \tag{3.27}$$

where $\tilde{W}_i$ is the horizontal lift of W_i, for $i = 1, 2, ...n$ and $r^i(t)$ (respectively $s^k(t)$) are the coefficients of H (respectively of X) in the representation using the base field $\tilde{W}_i$ restricted to σ.

Since π is a conformal submersion with horizontal distribution

$$\pi_*(\mathcal{H}(\nabla_H X)) = \nabla^*_{\pi_*(H)} \pi_* X + X(\phi)\pi_* H + H(\phi)\pi_* X - e^{2\phi}\pi_*(grad_\pi \phi) g_b(\pi_* X, \pi_* H).$$

Hence,

$$\begin{aligned}
\pi_*(\mathcal{H}(E^{'})) &= E^{'}_* + \pi_*(A_X U + A_X V + T_U V) - e^{2\phi}\pi_*(grad_\pi \phi) g_b(\pi_* X, \pi_* H) \\
&\quad + X(\phi)\pi_* H + H(\phi)\pi_* X.
\end{aligned}$$

Similarly we can prove $\mathcal{V}(E^{'}) = A_X H + T_U H + \mathcal{V}(V^{'})$. □

For $\sigma^{''}$ we have

Corollary 3.2. *Let σ be a curve in $\mathbf{M}$ with $X = \mathcal{H}(\sigma^{'})$ and $U = \mathcal{V}(\sigma^{'})$. Then,*

$$\begin{aligned} \pi_*(\mathcal{H}(\sigma^{''})) &= \sigma^{''}_* + \pi_*(2A_X U + T_U U) - e^{2\phi}\pi_*(grad_\pi \phi) g_b(\pi_* X, \pi_* X) \\ &\quad +2X(\phi)\pi_* X, && (3.28) \\ \mathcal{V}(\sigma^{''}) &= A_X X + T_U X + \mathcal{V}(U^{'}), && (3.29) \end{aligned}$$

where $\sigma^{''}_*$ denotes the covariant derivative of $(\pi \circ \sigma)^{'}$.

Now, for a conformal submersion with horizontal distribution we prove a necessary and sufficient condition for $\pi \circ \sigma$ to become a geodesic of $\mathbf{B}$ when σ is a geodesic of $\mathbf{M}$.

Theorem 3.6. *Let $\pi : (\mathbf{M}, \nabla, g_m) \to (\mathbf{B}, \nabla^*, g_b)$ be a conformal submersion with horizontal distribution. If σ is a geodesic of $\mathbf{M}$, then $\pi \circ \sigma$ is a geodesic of $\mathbf{B}$ if and only if*

$$\pi_*(2A_X U + T_U U)) + 2X(\phi)\pi_* X = \pi_*(grad_\pi \phi) \parallel X \parallel^2,$$

where $X = \mathcal{H}(\sigma^{'})$ and $U = \mathcal{V}(\sigma^{'})$ and $\parallel X \parallel^2 = g_m(X, X)$.

Proof. Since σ is a geodesic of $\mathbf{M}$ from (3.28) we get

$$\sigma^{''}_* = \pi_*(grad_\pi \phi) \parallel X \parallel^2 - \pi_*(2A_X U + T_U U) - 2d\phi(X)\pi_* X.$$

Hence, $\pi \circ \sigma$ is a geodesic of $\mathbf{B}$ if and only if

$$\pi_*(2A_X U + T_U U)) + 2X(\phi)\pi_* X = \pi_*(grad_\pi \phi) \parallel X \parallel^2 .$$

□

Remark 3.5. If σ is a horizontal geodesic (that is, σ is a geodesic with $\mathcal{V}(\sigma^{'}) = 0$), then $\pi \circ \sigma$ is a geodesic if and only if $2X(\phi)\pi_* X = \pi_*(grad_\pi \phi) \parallel X \parallel^2$.

Now we have,

Proposition 3.5. *Let $\pi : (\mathbf{M}, \nabla, g_m) \to (\mathbf{B}, \nabla^*, g_b)$ be a conformal submersion with horizontal distribution such that $A_Z Z = 0$ for all horizontal vectors Z. Then, every horizontal lift of a geodesic of $\mathbf{B}$ is a geodesic of $\mathbf{M}$ if and only if $2X(\phi)\pi_* X = \pi_*(grad_\pi \phi) \parallel X \parallel^2$, where X is the horizontal part of the tangent vector field of the horizontal lift of the geodesic of $\mathbf{B}$.*

Proof. Let α be a geodesic of $\mathbf{B}$ and σ be the horizontal lift of α. Then, $\pi \circ \sigma = \alpha$ and $\sigma^{'}(t) = \widetilde{(\alpha^{'}(t))}$. Let $X = \mathcal{H}(\sigma^{'}(t))$ and $U = \mathcal{V}(\sigma^{'}(t))$, clearly $X = \widetilde{(\alpha^{'}(t))}$ and $U = 0$. Then, from (3.28) and (3.29)

$$\begin{aligned}\pi_*(\mathcal{H}(\sigma^{''})) &= \alpha^{''} - \pi_*(grad_\pi \phi) \parallel X \parallel^2 + 2X(\phi)\pi_* X.\\ \mathcal{V}(\sigma^{''}) &= A_X X.\end{aligned}$$

Since, α is a geodesic and $A_X X = 0$ we have, $\sigma^{''} = 0$ if and only if $2X(\phi)\pi_* X = \pi_*(grad_\pi \phi) \parallel X \parallel^2$. That is, every horizontal lift of a geodesic of $\mathbf{B}$ is a geodesic of $\mathbf{M}$ if and only if $2X(\phi)\pi_* X = \pi_*(grad_\pi \phi) \parallel X \parallel^2$. □

Corollary 3.3. *Let $\pi : (\mathbf{M}, \nabla, g_m) \to (\mathbf{B}, \nabla^*, g_b)$ be a conformal submersion with horizontal distribution such that $A_Z Z = 0$ for all horizontal vectors Z. Then, ∇^* is geodesically complete if ∇ is geodesically complete and $2X(\phi)\pi_* X = \pi_*(grad_\pi \phi) \parallel X \parallel^2$, where X is the horizontal part of the tangent vector field of the horizontal lift of the geodesic of $\mathbf{B}$.*

Proof. Let α be a geodesic of $\mathbf{B}$ and $\tilde{\alpha}$ be its horizontal lift to $\mathbf{M}$, by Proposition (3.5) $\tilde{\alpha}$ is a geodesic of $\mathbf{M}$. Since ∇ is geodesically complete, $\tilde{\alpha}$ can be defined on the entire real line. Then, the projected curve of the extension of $\tilde{\alpha}$ is a geodesic and is the extension of α, that is ∇^* is geodesically complete. □

3.4 Affine and Conformal Submersions with Horizontal Distribution and Statistical Manifolds

In this section, we first discuss the affine submersion with horizontal distribution and statistical manifolds. A statistical structure is obtained on the manifold $\mathbf{B}$ induced by the affine submersion $\pi : \mathbf{M} \longrightarrow \mathbf{B}$ with the horizontal distribution $\mathcal{H}(\mathbf{M}) = \mathcal{V}^\perp(\mathbf{M})$. Abe and Hasegawa [21] obtained a necessary and sufficient condition for $(\mathbf{M}, \nabla, g_m)$ to become a statistical manifold for an affine submersion with horizontal distribution $\pi : (\mathbf{M}, \nabla) \longrightarrow (\mathbf{B}, \nabla^*)$. In the case of conformal submersion with horizontal distribution we obtained a necessary and sufficient condition for $(\mathbf{M}, \nabla, g_m)$ to become a statistical manifold. Also, we prove $\pi : (\mathbf{M}, \nabla) \longrightarrow (\mathbf{B}, \nabla^*)$ is a conformal submersion with horizontal distribution if and only if $\pi : (\mathbf{M}, \overline{\nabla}) \longrightarrow (\mathbf{B}, \overline{\nabla}^*)$ is a conformal submersion with horizontal distribution [22].

3.4.1 Affine Submersion with Horizontal Distribution and Statistical Manifolds

In this subsection, we discuss the affine submersion with horizontal distribution for statistical manifolds.

Definition 3.11. Let $\pi : (\mathbf{M}, \nabla) \longrightarrow (\mathbf{B}, \nabla^*)$ be an affine submersion with horizontal distribution $\mathcal{V}^{\perp}(\mathbf{M})$ and g be a semi-Riemannian metric on $\mathbf{M}$ and $\mathcal{H}(\nabla_{\tilde{X}}\tilde{Y})$ be projectable. Define the induced semi-Riemannian metric $\tilde{g}$ and the induced connection ∇' on $\mathbf{B}$ as

$$\begin{aligned} \tilde{g}(X, Y) &= g(\tilde{X}, \tilde{Y}), & (3.30) \\ \nabla'_X Y &= \pi_*(\nabla_{\tilde{X}}\tilde{Y}), & (3.31) \end{aligned}$$

where X, Y are in $\mathcal{X}(\mathbf{B})$.

Now, we show that $(\mathbf{B}, \nabla', \tilde{g})$ is a statistical manifold.

Theorem 3.7. *Let $(\mathbf{M}, \nabla, g)$ be a statistical manifold and $\pi : \mathbf{M} \longrightarrow \mathbf{B}$ be an affine submersion with horizontal distribution $\mathcal{H}(\mathbf{M}) = \mathcal{V}^{\perp}(\mathbf{M})$ and $\mathcal{H}(\nabla_{\tilde{X}}\tilde{Y})$ be projectable. Then, $(\mathbf{B}, \nabla', \tilde{g})$ is a statistical manifold.*

Proof. Let X, Y, Z be in $\mathcal{X}(\mathbf{B})$, then

$$\begin{aligned} (\nabla'_X \tilde{g})(Y, Z) &= X\tilde{g}(Y, Z) - \tilde{g}(\nabla'_X Y, Z) - \tilde{g}(Y, \nabla'_X Z) \\ &= \tilde{X}g(\tilde{Y}, \tilde{Z}) - g(\nabla_{\tilde{X}}\tilde{Y}, \tilde{Z}) - g(\tilde{Y}, \nabla_{\tilde{X}}\tilde{Z}) \\ &= (\nabla_{\tilde{X}}g)(\tilde{Y}, \tilde{Z}). \end{aligned}$$

Since $(\mathbf{M}, \nabla, g)$ is a statistical manifold, $(\mathbf{B}, \nabla', \tilde{g})$ is also a statistical manifold. $\square$

Let $(\mathbf{M}, g_m)$ be a semi-Riemannian manifold with affine connection ∇ and $\overline{\nabla}$ denotes the conjugate connection of ∇ with respect to g_m. Let $(\mathbf{B}, g_b)$ be another semi-Riemannian manifold with affine connection ∇^* and $\overline{\nabla^*}$ denotes the conjugate connection of ∇^* with respect to g_b.

Proposition 3.6. *[21] Let $\pi : (\mathbf{M}, g_m) \longrightarrow (\mathbf{B}, g_b)$ be a semi-Riemannian submersion. Then, $\pi : (\mathbf{M}, \nabla) \longrightarrow (\mathbf{B}, \nabla^*)$ is an affine submersion with horizontal distribution $\mathcal{H}(\mathbf{M}) = \mathcal{V}(\mathbf{M})^{\perp}$ if and only if $\pi : (\mathbf{M}, \overline{\nabla}) \longrightarrow (\mathbf{B}, \overline{\nabla^*})$ is an affine submersion with the same horizontal distribution.*

Proof. Let $\tilde{X}, \tilde{Y}$ and $\tilde{Z}$ be the horizontal lift of vector fields X, Y and Z on $\mathbf{B}$. Now,

$$\begin{aligned} \tilde{X} g_m(\tilde{Y}, \tilde{Z}) &= X g_b(Y, Z) \\ &= g_b(\nabla^*_X Y, Z) + g_b(Y, \overline{\nabla}^*_X Z). \end{aligned} \tag{3.32}$$

On the other hand,

$$\begin{aligned} \tilde{X} g_m(\tilde{Y}, \tilde{Z}) &= g_m(\nabla_{\tilde{X}} \tilde{Y}, \tilde{Z}) + g_b(\tilde{Y}, \overline{\nabla}_{\tilde{X}} \tilde{Z}) \\ &= gb(\pi_*(\nabla_{\tilde{X}} \tilde{Y}), Z) + g_b(Y, \pi_*(\overline{\nabla}_{\tilde{X}} \tilde{Z})). \end{aligned} \tag{3.33}$$

Now, from the equations (3.32) and (3.33) we get

$$g_b(\pi_*(\nabla_{\tilde{X}} \tilde{Y}) - \nabla^*_X Y, Z) = g_b(Y, \overline{\nabla}^*_X Z - \pi_*(\overline{\nabla}_{\tilde{X}} \tilde{Z})).$$

Hence, $\pi : (\mathbf{M}, \nabla) \longrightarrow (\mathbf{B}, \nabla^*)$ is an affine submersion with horizontal distribution $\mathcal{H}(\mathbf{M}) = \mathcal{V}(\mathbf{M})^\perp$ if and only if $\pi : (\mathbf{M}, \overline{\nabla}) \longrightarrow (\mathbf{B}, \overline{\nabla^*})$ is an affine submersion with the same horizontal distribution. $\square$

Lemma 3.1. *[21] Let $\pi : (\mathbf{M}, g_m) \longrightarrow (\mathbf{B}, g_b)$ be a semi-Riemannian submersion and $\pi : (\mathbf{M}, \nabla) \longrightarrow (\mathbf{B}, \nabla^*)$ be an affine submersion with horizontal distribution $\mathcal{V}(\mathbf{M})^\perp$, then for horizontal vectors X, Y and vertical vectors U, V, W,*

$$(\nabla_{\tilde{X}} g_m)(\tilde{X}_1, \tilde{X}_2) = (\nabla^*_X g_b)(X_1, X_2) \circ \pi, \tag{3.34}$$
$$(\nabla_V g_m)(X, Y) = -g_m(S_V X, Y), \tag{3.35}$$
$$(\nabla_X g_m)(V, Y) = -g_m(A_X V, Y) + g_m(\overline{A}_X V, Y), \tag{3.36}$$
$$(\nabla_X g_m)(V, W) = -g_m(S_X V, W), \tag{3.37}$$
$$(\nabla_V g_m)(X, W) = -g_m(T_V X, W) + g_m(\overline{T}_V X, W), \tag{3.38}$$
$$(\nabla_U g_m)(V, W) = (\hat{\nabla}_U \hat{g}_m)(V, W), \tag{3.39}$$

where $\tilde{X}_i$ are the horizontal lift of vector fields X_i on $\mathbf{B}$, $\hat{g}$ is the induced metric on the fibers and $S_V X = \nabla_V X - \overline{\nabla}_V X$.

Proof. Now,

$$\begin{aligned} (\nabla_{\tilde{X}} g_m)(\tilde{X}_1, \tilde{X}_2) &= \tilde{X} g_m(\tilde{X}_1, \tilde{X}_2) - g_m(\nabla_{\tilde{X}} \tilde{X}_1, \tilde{X}_2) - g_m(\tilde{X}_1, \nabla_{\tilde{X}} \tilde{X}_2) \\ &= X g_b(X_1, X_2) - g_b(\pi_*(\nabla_{\tilde{X}} \tilde{X}_1), X_2) \\ &\quad - g_b(X_1, \pi_*(\nabla_{\tilde{X}} \tilde{X}_2)) \end{aligned}$$

$$= \quad Xg_b(X_1, X_2) - g_b(\nabla^*_X X_1, X_2) - g_b(X_1, \nabla^*_X X_2)$$
$$= \quad (\nabla^*_X g_b)(X_1, X_2) \circ \pi.$$

Similarly, we can prove the other equations. □

Using the lemma (3.1), Abe and Hasegawa proved the following theorem.

Theorem 3.8. *[21] Assume that $Tor(\nabla) = 0$. Let $\pi : (\mathbf{M}, g_m) \longrightarrow (\mathbf{B}, g_b)$ be a semi-Riemannian submersion and $\pi : (\mathbf{M}, \nabla) \longrightarrow (\mathbf{B}, \nabla^*)$ be an affine submersion with horizontal distribution $\mathcal{H}(\mathbf{M}) = \mathcal{V}^{\perp}(\mathbf{M})$. Then, $(\mathbf{M}, \nabla, g_m)$ is a statistical manifold if and only if*

1. $\mathcal{H}(S_V X) = A_X V - \overline{A}_X V$.
2. $\mathcal{V}(S_X V) = T_V X - \overline{T}_V X$.
3. $(\pi^{-1}(b), \hat{\nabla}^b, \hat{g}^b_m)$ *is a statistical manifold for each* $b \in \mathbf{B}$.
4. $(\mathbf{B}, \nabla^*, g_b)$ *is a statistical manifold.*

Proof. Suppose $(\mathbf{M}, \nabla, g_m)$ is a statistical manifold, then ∇g_m is symmetric.
So $(\nabla_V g_m)(X, Y) = (\nabla_X g_m)(V, Y)$, where X, Y are horizontal vector fields and V is a vertical vector field. Then, from (3.35) and (3.36) of the above lemma $g_m(S_V X, Y) = g_m(A_X V, Y) - g_m(\overline{A}_X V, Y)$. This implies, $\mathcal{H}(S_V X) = A_X V - \overline{A}_X V$. Similarly from (3.37) and (3.38) of the above lemma, we have $\mathcal{V}(S_X V) = T_V X - \overline{T}_V X$.

Since ∇g_m is symmetric, from (3.39) of the above lemma, we get $\hat{\nabla}^b \hat{g}^b$ is symmetric, so $(\pi^{-1}(b), \hat{\nabla}^b, \hat{g}^b_m)$ is a statistical manifold. Also from (3.34) of the above lemma, we get $(\nabla_{\tilde{X}} g_m)(\tilde{X}_1, \tilde{X}_2) = (\nabla^*_X g_b)(X_1, X_2)$, where $\tilde{X}_i$ are the horizontal lift of the vector fields X_i on $\mathbf{B}$. Since, ∇g_m is symmetric $\nabla^* g_b$ is also symmetric. Hence, $(\mathbf{B}, \nabla^*, g_b)$ is a statistical manifold.

Conversely, if all the four conditions hold then from the above lemma
$\nabla_E g_m(F, G) = \nabla_F g_m(E, G)$, for E, F and G in $\mathcal{X}(\mathbf{M})$. That is, ∇g_m is symmetric on $\mathbf{M}$ and hence $(\mathbf{M}, \nabla, g_m)$ is a statistical manifold. □

3.4.2 Conformal Submersion with Horizontal Distribution and Statistical Manifolds

In this subsection, we discuss the conformal submersion with horizontal distribution for statistical manifolds [22].

Let $(\mathbf{M}, g_m)$ and $(\mathbf{B}, g_b)$ be semi-Riemannian manifolds with affine connections ∇ and ∇^*, respectively. First, we prove the following proposition.

Proposition 3.7. *Let $\pi : (\mathbf{M}, g_m) \longrightarrow (\mathbf{B}, g_b)$ be a conformal submersion. Then, $\pi : (\mathbf{M}, \nabla) \longrightarrow (\mathbf{B}, \nabla^*)$ is a conformal submersion with horizontal distribution if and only if $\pi : (\mathbf{M}, \overline{\nabla}) \longrightarrow (\mathbf{B}, \overline{\nabla^*})$ is a conformal submersion with horizontal distribution.*

Proof. Now,

$$\begin{aligned}\tilde{X} g_m(\tilde{Y}, \tilde{Z}) &= 2e^{2\phi} d\phi(\tilde{X}) g_b(Y, Z) + e^{2\phi} X g_b(Y, Z) \\ &= 2e^{2\phi} d\phi(\tilde{X}) g_b(Y, Z) + e^{2\phi}\{g_b(\nabla^*_X Y, Z) + g_b(Y, \overline{\nabla^*}_X Z)\}\end{aligned}$$

and

$$\begin{aligned}\tilde{X} g_m(\tilde{Y}, \tilde{Z}) &= g_m(\nabla_{\tilde{X}} \tilde{Y}, \tilde{Z}) + g_m(\tilde{Y}, \overline{\nabla}_{\tilde{X}} \tilde{Z}) \\ &= e^{2\phi} g_b(\pi_*(\nabla_{\tilde{X}} \tilde{Y}), Z) + e^{2\phi} g_b(Y, \pi_*(\overline{\nabla}_{\tilde{X}} \tilde{Z})).\end{aligned} \tag{3.40}$$

Since,

$$\begin{aligned}g_b(\pi_*(\nabla_{\tilde{X}} \tilde{Y}), Z) &= g_b(\nabla^*_X Y, Z) - d\phi(\tilde{Z}) g_b(X, Y) \\ &\quad + \{d\phi(\tilde{X}) g_b(Y, Z) + d\phi(\tilde{Y}) g_b(Z, X)\}.\end{aligned} \tag{3.41}$$

From (3.40) and (3.41) we get

$$\begin{aligned}g_b(\pi_*(\overline{\nabla}_{\tilde{X}} \tilde{Z}), Y) &= g_b(\overline{\nabla^*}_X Z, Y) - d\phi(\tilde{Y}) g_b(X, Z) \\ &\quad + \{d\phi(\tilde{X}) g_b(Y, Z) + d\phi(\tilde{Z}) g_b(X, Y)\}.\end{aligned}$$

Hence, $\pi : (\mathbf{M}, \overline{\nabla}) \longrightarrow (\mathbf{B}, \overline{\nabla^*})$ is a conformal submersion with horizontal distribution. Converse is obtained by interchanging ∇, ∇^* with $\overline{\nabla}$, $\overline{\nabla^*}$ in the above proof. □

Lemma 3.2. *Let $\pi : (\mathbf{M}, g_m) \longrightarrow (\mathbf{B}, g_b)$ be a conformal submersion and $\pi : (\mathbf{M}, \nabla) \longrightarrow$*

$(\mathbf{B}, \nabla^*)$ *be a conformal submersion with horizontal distribution* $\mathcal{V}(\mathbf{M})^\perp$*, then*

$$\begin{aligned}
(\nabla_{\tilde{X}} g_m)(\tilde{X}_1, \tilde{X}_2) &= e^{2\phi}(\nabla^*_X g_b)(X_1, X_2), && (3.42)\\
(\nabla_V g_m)(X, Y) &= -g_m(S_V X, Y), && (3.43)\\
(\nabla_X g_m)(V, Y) &= -g_m(A_X V, Y) + g_m(\overline{A}_X V, Y), && (3.44)\\
(\nabla_X g_m)(V, W) &= -g_m(S_X V, W), && (3.45)\\
(\nabla_V g_m)(X, W) &= -g_m(T_V X, W) + g_m(\overline{T}_V X, W), && (3.46)\\
(\nabla_U g_m)(V, W) &= (\hat{\nabla}_U \hat{g}_m)(V, W), && (3.47)
\end{aligned}$$

for horizontal vector fields X, Y *on* $\mathbf{M}$ *and vertical vector fields* U, V, W *on* $\mathbf{M}$*.* $\tilde{X}_i$ *are the horizontal lift of vector fields* X_i *on* $\mathbf{B}$*,* $\hat{g}$ *is the induced metric on the fibers and* $S_V X = \nabla_V X - \overline{\nabla}_V X$*.*

Proof. Now,

$$\begin{aligned}
(\nabla_{\tilde{X}} g_m)(\tilde{X}_1, \tilde{X}_2) &= \tilde{X} g_m(\tilde{X}_1, \tilde{X}_2) - g_m(\nabla_X \tilde{X}_1, \tilde{X}_2) - g_m(\tilde{X}_1, \nabla_X \tilde{X}_2)\\
&= \tilde{X} e^{2\phi} g_b(X_1, X_2) - e^{2\phi} g_b(\pi_*(\nabla_{\tilde{X}} \tilde{X}_1), X_2)\\
&\quad -e^{2\phi} g_b(X_1, \pi_*(\nabla_{\tilde{X}} \tilde{X}_2))\\
&= 2e^{2\phi} d\phi(\tilde{X}) g_b(X_1, X_2) + e^{2\phi} X g_b(X_1, X_2)\\
&\quad -e^{2\phi} g_b(\pi_*(\nabla_{\tilde{X}} \tilde{X}_1), X_2) - e^{2\phi} g_b(X_1, \pi_*(\nabla_{\tilde{X}} \tilde{X}_2)).
\end{aligned}$$

Since

$$\begin{aligned}
g_b(\pi_*(\nabla_{\tilde{X}} \tilde{X}_i), X_j) &= g_b(\nabla^*_X X_i, X_j) - d\phi(\tilde{X}_j) g_b(X, X_i)\\
&\quad +\{d\phi(\tilde{X}) g_b(X_i, X_j) + d\phi(\tilde{X}_i) g_b(X_j, X)\},
\end{aligned}$$

where $i, j = 1, 2$ and $i \neq j$, we get

$$(\nabla_{\tilde{X}} g_m)(\tilde{X}_1, \tilde{X}_2) = e^{2\phi}(\nabla^*_X g_b)(X_1, X_2).$$

Similarly, we can prove the other equations. □

Now, we prove a necessary and sufficient condition for $(\mathbf{M}, \nabla, g_m)$ to be a statistical manifold for a conformal submersion with horizontal distribution.

Theorem 3.9. *Let* $\pi : (\mathbf{M}, g_m) \longrightarrow (\mathbf{B}, g_b)$ *be a conformal submersion and* $\pi : (\mathbf{M}, \nabla) \longrightarrow (\mathbf{B}, \nabla^*)$ *be a conformal submersion with horizontal distribution* $\mathcal{H}(\mathbf{M}) = \mathcal{V}^\perp(\mathbf{M})$ *and* ∇

be torsion-free. Then, $(\mathbf{M}, \nabla, g_m)$ *is a statistical manifold if and only if*

1. $\mathcal{H}(S_V X) = A_X V - \overline{A}_X V.$

2. $\mathcal{V}(S_X V) = T_V X - \overline{T}_V X.$

3. $(\pi^{-1}(b), \hat{\nabla}^b, \hat{g}^b_m)$ *is a statistical manifold for each* $b \in \mathbf{b}$.

4. $(\mathbf{B}, \nabla^*, g_b)$ *is a statistical manifold.*

Proof. Suppose $(\mathbf{M}, \nabla, g_m)$ is a statistical manifold, then ∇g_m is symmetric. So $(\nabla_V g_m)(X, Y)$ = $(\nabla_X g_m)(V, Y)$, where X, Y are horizontal vector fields and V is a vertical vector field. Then, from (3.43) and (3.44) of the above lemma $g_m(S_V X, Y) = g_m(A_X V, Y) - g_m(\overline{A}_X V, Y)$. This implies, $\mathcal{H}(S_V X) = A_X V - \overline{A}_X V$. Similarly, from (3.45) and (3.46) of the above lemma, we have $\mathcal{V}(S_X V) = T_V X - \overline{T}_V X$.

Since ∇g_m is symmetric, from (3.47) of the above lemma, we get $\hat{\nabla}^b \hat{g}^b$ is symmetric, so $(\pi^{-1}(b), \hat{\nabla}^b, \hat{g}^b_m)$ is a statistical manifold. Also from (3.42) of the above lemma, we get $(\nabla_{\tilde{X}} g_m)(\tilde{X}_1, \tilde{X}_2) = e^{2\phi}(\nabla^*_X g_b)(X_1, X_2)$, where $\tilde{X}_i$ are the horizontal lift of the vector fields X_i on $\mathbf{B}$. Since, ∇g_m is symmetric $\nabla^* g_b$ is also symmetric. Hence, $(\mathbf{B}, \nabla^*, g_b)$ is a statistical manifold.

Conversely, if all the four conditions hold then from the above lemma $\nabla_E g_m(F, G) = \nabla_F g_m(E, G)$, for vector fields E, F and G on $\mathbf{M}$. That is, ∇g_m is symmetric on $\mathbf{M}$ and hence $(\mathbf{M}, \nabla, g_m)$ is a statistical manifold. $\square$

Chapter 4

Tangent Bundles and Harmonic Maps of Statistical Manifolds

Geometry of tangent bundles is a fruitful domain of differential geometry that gives a clear insight into the classical theory and provides many new problems in the area of differential geometry. Let M be a Riemannian manifold and $\pi : T\mathrm{M} \longrightarrow \mathrm{M}$ be the tangent map. Matsuzoe and Inoguchi [23], have shown that the tangent bundle $T\mathrm{M}$ has got various statistical manifold structures using complete, vertical and horizontal lifts of metric, connection and the cubic form. In [24], Balan et al. have proved that $(T\mathrm{M}, \tilde{\nabla}, g^s)$ is a statistical manifold, where g^s is the Sasaki lift metric.

Harmonic mapping provides a natural way of mapping two manifolds by minimizing distortion induced by the mapping. A systematic study of harmonic maps was initiated by Eells and Sampson [51]. Presently, we see an increasing interest in harmonic maps between statistical manifolds [26], [27]. In [26], Uohashi obtained a condition for the harmonicity on α-conformally equivalent statistical manifolds. Oproiu [30] obtained conditions for the tangent map to be harmonic in the case of tangent bundles equipped with the metrics obtained from the complete lift of metrics and the vertical lift of an appropriate tensor field.

In this chapter, statistical manifold structures on tangent bundles and harmonic maps between statistical manifolds and tangent bundles were discussed. Note that the tangent map can be considered as a submersion and we used the results in the geometry of submersions to obtain geometric structures on tangent bundles. In Section 4.1, we prove a necessary and sufficient condition for $T\mathrm{M}$ to become a statistical manifold with the complete lift connection and the Sasaki lift metric. In Section 4.2, we first look at the definition of the harmonic map using tension field. Then, prove a necessary and sufficient condition for the harmonicity of the identity map for conformally-projectively equivalent statistical manifolds. The conformal statistical submerion is defined which is a generalization of the statistical submersion and proved that harmonicity and conformality cannot coexist [28]. In Section 4.3, for statistical manifolds we proved that a smooth map $\phi : \mathbf{M} \longrightarrow \mathbf{B}$ is harmonic with respect to ∇ and ∇^* if and only if it is harmonic with respect to the conjugate connections $\overline{\nabla}$ and $\overline{\nabla^*}$. Then, given a necessary condition for the harmonicity of the tangent map with respect to the complete lift structure on the tangent bundles. Also, prove $\pi : (\mathbf{M}, \nabla, g_m) \longrightarrow (\mathbf{B}, \nabla^*, g_b)$ is a statistical submersion if and only if $\pi_* : (T\mathbf{M}, \nabla^c, g_m^c) \longrightarrow (T\mathbf{B}, \nabla^{*c}, g_b^c)$ is a statistical submersion.

4.1 Statistical Structures on Tangent Bundle

In this section, we discuss the work of Matsuzoe and Inoguchi [23] and Balan et al. [24] on obtaining the various statistical manifold structures on $T\mathbf{M}$. Then, proved a necessary and sufficient condition for $T\mathbf{M}$ to become a statistical manifold with the complete lift connection and the Sasaki lift metric [22].

4.1.1 Vertical, Complete and Horizontal Lifts on Tangent Bundles.

In this subsection, we first look at the concept of vertical, complete and horizontal lifts of vector fields, functions, tensors, metrics and connections. Then, various ways of getting the statistical manifold structures on the tangent bundle $T\mathbf{M}$ are discussed [23], [24].

Let $\mathbf{M}$ be an n-dimensional manifold and $T\mathbf{M}$ denote the tangent bundle on $\mathbf{M}$, $\pi : T\mathbf{M} \longrightarrow \mathbf{M}$ be the natural projection defined by $X_x \in T_x\mathbf{M} \longrightarrow x \in \mathbf{M}$. Taking a local coordinate system $(U; x^1, ..., x^n)$ on $\mathbf{M}$, the induced coordinate system on $\pi^{-1}(U)$ is denoted by $(x^1, ..x^n; u^1, ..u^n)$. Let $(x; u)$ be a point on $T\mathbf{M}$, denote the kernel of $\pi_{*(x;u)} : T_{(x;u)}(T\mathbf{M}) \longrightarrow T_x\mathbf{M}$ by $\mathcal{V}_{(x;u)}(T\mathbf{M})$ called the vertical subspace of $T_{(x;u)}(T\mathbf{M})$ at $(x; u)$. Note that the vertical subspace $\mathcal{V}_{(x;u)}(T\mathbf{M})$ is spanned by $\{\frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^2}, \dots \frac{\partial}{\partial u^n}\}$. The two linear spaces $T_x\mathbf{M}$ and $\mathcal{V}_{(x;u)}(T\mathbf{M})$ have the same dimension, so there is a canonical linear isomorphism $v : T_x\mathbf{M} \longrightarrow \mathcal{V}_{(x;u)}(T\mathbf{M})$ called the vertical lift. That is, for any tangent vector X_x on $\mathbf{M}$ with local expression $X_x = X_x^i \frac{\partial}{\partial x^i} \mid_x$, the vertical lift X_x^v to $T\mathbf{M}$ is defined by $X_x^v = X_x^i \frac{\partial}{\partial u^i} \mid_u$. This definition is independent of the choice of the local coordinate system.

Definition 4.1. Let $f : \mathbf{M} \longrightarrow \mathbf{R}$ be a smooth function on $\mathbf{M}$ and $\pi : T\mathbf{M} \longrightarrow \mathbf{M}$ be the natural projection. The vertical lift of f is denoted by f^v and is defined as $f^v = f \circ \pi$. For a vector field $X = X^i \frac{\partial}{\partial x^i}$ on $\mathbf{M}$ the vertical lift is denoted by X^v and defined as $X^v = (X^i)^v \frac{\partial}{\partial u^i}$.

Note. The definition of vertical lift of a vector field is independent of the choice of local coordinate system. Note that, for any $X, Y \in \mathcal{X}(\mathbf{M})$, $[X^v, Y^v] = 0$.

Let f be a smooth function on $\mathbf{M}$. Then, the vertical lift of df is defined by $(df)^v = d(f^v)$, in particular for local coordinate functions x^i, $(dx^i)^v = d(x^i)^v$. The vertical lift of a 1-form ω with local expression $\omega = \omega_i dx^i$ is defined as $\omega^v = (\omega_i)^v d(x^i)^v$. The definition of ω^v is independent of the local coordinate system.

The vertical lift operation can be extended to the whole tensor algebra $\mathcal{T}(\mathbf{M})$ using the law $(P \otimes Q)^v = P^v \otimes Q^v$ for any tensor fields P and Q on $\mathbf{M}$.

Now we discuss the complete lift operation to $T\mathbf{M}$. [52, 53, 54]

Definition 4.2. Let $f : \mathbf{M} \longrightarrow \mathbf{R}$ be a smooth map, the complete lift f^c of f on $T\mathbf{M}$ is defined as $f^c = i(df) = u^i \frac{\partial f}{\partial x^i}$. The complete lift X^c on $T\mathbf{M}$ of the vector field X on $\mathbf{M}$ is characterized by the formula $X^c(f^c) = (Xf)^c$, for all $f \in C^\infty(\mathbf{M})$. In local coordinates, the complete lift X^c of $X = X^i \frac{\partial}{\partial x^i}$ has the local expression

$$X^c \quad = \quad (X^i)^v \frac{\partial}{\partial x^i} + u^j \frac{\partial X^i}{\partial x^j} \frac{\partial}{\partial u^i}.$$

Remark 4.1. From the above formula note that for any point $u = (x; u)$ in $T\mathbf{M}$ other than zero, the set $\{X^c_u \mid X \in \mathcal{X}(\mathbf{M})\}$ is the whole tangent space of $T\mathbf{M}$ at u.

Definition 4.3. The complete lift to 1-from ω is defined as

$$\omega^c(X^c) \quad = \quad (\omega(X))^c.$$

More generally, the complete lift to the full tensor algebra $\mathcal{T}(\mathbf{M})$ is given by the rule

$$(P \otimes Q)^c = P^c \otimes Q^v + P^v \otimes Q^c,$$

for tensor fields P and Q on $\mathbf{M}$.

Now, we state certain important formulae related to the vertical and the complete lifts [55].

1. Let P be a tensor field of type (r, s), $r = 0, 1$ on $\mathbf{M}$. Then,

$$P^c(X^c_1, \cdots, X^c_s) = (P(X_1, \cdots, X))^c, \;\; P^c(X^v_1, \cdots, X^v_s) = 0. \tag{4.1}$$

2. Let g be a tensor field of type $(0,2)$ on $\mathbf{M}$. Then,

$$g^c(X^c, Y^c) = g(X,Y)^c, \;\; g^c(X^c, Y^v) = g(X,Y)^v, \;\; g^c(X^v, Y^v) = 0. \tag{4.2}$$

In particular, if g is a semi-Riemannian metric on $\mathbf{M}$ then g^c is a semi-Riemannian metric on $T\mathbf{M}$. The metric g^c is called the complete lift metric on $T\mathbf{M}$.

Now, we discuss the complete lift operation of a linear connection.

Definition 4.4. Let ∇ be a linear connection on $\mathbf{M}$, then the complete lift ∇^c on $T\mathbf{M}$ is defined as $\nabla^c_{X^c} Y^c = (\nabla_X Y)^c$, for every $X, Y \in \mathcal{X}(\mathbf{M})$.

Proposition 4.1. *[23] Let $(\mathbf{M}, \nabla)$ be a manifold with linear connection ∇. Let T and R be the torsion and the curvature tensor of ∇, respectively. Then, the torsion and the curvature of ∇^c are T^c and R^c, respectively.*

Proof. By definition of complete lift $T^c(X^c, Y^c) = (T(X,Y))^c$. Since,

$$\begin{aligned} (T(X,Y))^c &= (\nabla_X Y - \nabla_Y X - [X,Y])^c, \\ &= \nabla^c_{X^c} Y^c - \nabla^c_{Y^c} X^c - [X^c, Y^c], \\ &= Tor\nabla^c(X^c, Y^c), \end{aligned}$$

where the $Tor\nabla^c$ is the torsion of ∇^c. Then,

$$T^c(X^c, Y^c) = Tor\nabla^c(X^c, Y^c).$$

That is, torsion of ∇^c is T^c. Similarly the curvature relation can be proved. $\square$

Proposition 4.2. *[23] Let $(\mathbf{M}, \nabla, g)$ be a statistical manifold. Then $(T\mathbf{M}, \nabla^c, g^c)$ is a statistical manifold, moreover the conjugate connection of ∇^c is $\overline{(\nabla^c)} = (\overline{\nabla})^c$.*

Proof. Since ∇ is torsion free, ∇^c also torsion free by proposition (4.1). Now, by the definition of the complete lift

$$(\nabla^c_{X^c} g^c)(Y^c, Z^c) = \{(\nabla_X g)(Y,Z)\}^c = \{(\nabla_Y g)(X,Z)\}^c = (\nabla^c_{Y^c} g^c)(X^c, Z^c).$$

Hence $\nabla^c g^c$ is symmetric. That is, $(T\mathbf{M}, \nabla^c, g^c)$ is a statistical manifold. Now,

$$\begin{aligned} g^c(Y^c, \overline{(\nabla^c)}_{X^c} Z^c) &= X^c g^c(Y^c, Z^c) - g^c(\nabla^c_{X^c} Y^c, Z^c), \\ &= X^c \{g(Y, Z)\}^c - \{g(\nabla_X Y, Z)\}^c, \\ &= \{X g(Y, Z) - g(\nabla_X Y, Z)\}^c = \{g(Y, \overline{\nabla}_X Z)\}^c, \\ &= g^c(Y^c, (\overline{\nabla})^c_{X^c} Z^c). \end{aligned}$$

Hence $\overline{(\nabla^c)} = (\overline{\nabla})^c$. □

Now, we look at the horizontal lifts on the tangent bundle. Let $\mathbf{M}$ be a smooth n-dimensional manifold and ∇ be a torsion free linear connection on $\mathbf{M}$. The vertical subspace $\mathcal{V}_{(x;u)}(T\mathbf{M})$ of $T_{(x;u)}(T\mathbf{M})$ at $(x; u)$ defines a smooth distribution $\mathcal{V}$ on $T\mathbf{M}$ called the vertical distribution. Also, there exists a smooth distribution $x \longrightarrow \mathcal{H}(T\mathbf{M})_x$ depending on the linear connection ∇ such that

$$T_{(x;u)}(T\mathbf{M}) = \mathcal{H}(T\mathbf{M})_x \oplus \mathcal{V}_{(x;u)}(T\mathbf{M}).$$

This distribution is called the horizontal distribution.

Definition 4.5. Let X be a vector field on $\mathbf{M}$, then the horizontal lift of X on $T\mathbf{M}$ is the unique vector field X^h on $T\mathbf{M}$ such that $\pi_*(X^h_{(x;u)}) = X_{\pi((x;u))}$ for all $(x; u) \in T\mathbf{M}$. In local coordinates if $X = X^i \frac{\partial}{\partial x^i}$, then

$$X^h = (X^i)^v \frac{\partial}{\partial x^i} - (X^j)^v u^k \Gamma^i_{j,k} \frac{\partial}{\partial u^i}.$$

Here $\Gamma^i_{j,k}$ is the connection coefficient of ∇.

Definition 4.6. Let g be a semi-Riemannian metric on $\mathbf{M}$, then the horizontal lift g^h on $\mathbf{M}$ is defined as $g^h(X^h, Y^h) = g^h(X^v, Y^v) = 0$ and $g^h(X^h, Y^v) = g(X, Y)$, for $X, Y \in \mathcal{X}(\mathbf{M})$. The horizontal lift ∇^h on $T\mathbf{M}$ of linear connection ∇ on $\mathbf{M}$ is defined as $\nabla^h_{X^v} Y^v = 0$, $\nabla^h_{X^v} Y^h = 0$, $\nabla^h_{X^h} Y^v = (\nabla_X Y)^v$, $\nabla^h_{X^h} Y^h = (\nabla_X Y)^h$, for $X, Y \in \mathcal{X}(\mathbf{M})$.

Remark 4.2. Note that g^h can be characterized by

$$g^h_{(x;u)}(\tilde{X}, \tilde{Y}) = g_x(\pi_* \tilde{X}, \mathcal{K}\tilde{Y}) + g_x(\mathcal{K}\tilde{X}, \pi_* \tilde{Y}). \tag{4.3}$$

Here the map $\mathcal{K} : T(T\mathbf{M}) \longrightarrow T\mathbf{M}$ is defined by

$$\mathcal{K}_u X^h_u = 0, \quad \mathcal{K}_u X^v_u = X_x,$$

called the connection map [56]. Also, note that even if ∇ is torsion-free its horizontal lift ∇^h may have non trivial torsion.

Definition 4.7. Let g be a semi-Riemannian metric on $(\mathbf{M}, \nabla)$. The semi-Riemannian metric g^s on $T\mathbf{M}$ is defined as, $g^s_{(x;u)}(X^h, Y^h) = g_x(X, Y)$, $g^s_{(x;u)}(X^h, Y^v) = 0$, $g^s_{(x;u)}(X^v, Y^v)$ = $g_x(X, Y)$. The metric g^s is called the Sasaki lift metric.

Note that the Sasaki lift metric g^s has the following representation.

$$g^s_{(x;u)}(\tilde{X}, \tilde{Y}) = g_x(\pi_*\tilde{X}, \pi_*\tilde{Y}) + g_x(\mathcal{K}\tilde{X}, \mathcal{K}\tilde{Y}). \tag{4.4}$$

Remark 4.3. In [55], Yano and Ishihara introduced the γ operator for defining the horizontal lift from the complete lift. Let X be a vector field on $\mathbf{M}$, with local expression $X = X^i\frac{\partial}{\partial x^i}$, $\nabla X = X^i_j\frac{\partial}{\partial x^i} \otimes dx^j$, where $X^i_j = \frac{\partial X^i}{\partial x^j} + X^k\Gamma^i_{j,k}$. Define $\gamma(\nabla X) = u^j X^i_j \frac{\partial}{\partial u^i}$ with respect to the induced coordinate $(x^1, ..., x^n; u^1, ..., u^n)$. Then, $X^h = X^c - \gamma(\nabla X)$, note that $\gamma(\nabla X)$ is the vertical part of X^c.

The horizontal lift operation can also be extended to $\mathcal{T}(\mathbf{M})$ by the rule

$$(P \otimes Q)^h = P^v \otimes Q^h + P^h \otimes Q^v,$$

for tensor fields P and Q on $\mathbf{M}$. Now, we discuss certain properties of horizontal lift operations connecting covariant derivatives of tensor fields [55]. Let $(\mathbf{M}, \nabla)$ be a manifold with affine connection then for $X \in \mathcal{X}(\mathbf{M})$

1. $\nabla^h_{X^c} P^v = (\nabla_X P)^v$, $\nabla^h_{X^c} P^h = (\nabla_X P)^h$.

2. $\nabla^h_{X^v} P^v = 0$, $\nabla^h_{X^v} P^h = 0$.

Remark 4.4. As a consequence of the above properties, Matsuzoe and Inoguchi [23] proved that if $(\mathbf{M}, \nabla, g)$ is a statistical manifold, then $(T\mathbf{M}, \nabla^h, g^s)$ or $(T\mathbf{M}, \nabla^h, g^h)$ is a statistical manifold if and only if $\nabla g = 0$.

Definition 4.8. Let $(\mathbf{M}, \nabla, g)$ be a statistical manifold. Define the tensor field K of type-$(1, 2)$ by

$$g(K(X)Y, Z) = C(X, Y, Z) = \nabla_X g(Y, Z),$$

for X, Y, Z in $\mathcal{X}(\mathbf{M})$. This tensor field K is called the skewness operator of $(\mathbf{M}, \nabla, g)$. Since C is symmetric, $K(X)$ is symmetric with respect to g.

Note. The difference tensor defined in Chapter 2 and the skewness operator defined above are same. For, consider

$$K_X Y = \overline{\nabla}_X Y - \nabla_X Y.$$

Since $\overset{g}{\nabla} = \frac{\nabla + \overline{\nabla}}{2}$,

$$K_X Y = 2(\overset{g}{\nabla}_X Y - \nabla_X Y).$$

Then $K_X = 2(\overset{g}{\nabla}_X - \nabla_X)$ is a tensor of type $(1,1)$ that maps $Y \in T_x\mathbf{M}$ to $K_X Y \in T_x\mathbf{M}$. Since $\overset{g}{\nabla}_X g = 0$, we get

$$\begin{aligned}(\nabla_X g)(Y,Z) &= -\frac{1}{2}(K_X g)(Y,Z) \\ &= \frac{1}{2}(g(K_X Y, Z) + g(Y, K_X Z)).\end{aligned} \tag{4.5}$$

We know that $(\nabla_X g)(Y,Z)$ is symmetric in X, Y and Z and that $g(K_X Y, Z)$ is symmetric in X and Y. This implies $g(Y, K_X Z))$ is symmetric in X and Y. Since ∇ and $\overline{\nabla}$ are torsion free, we get $g(Y, K_X Z))$ is symmetric in X, Z, and therefore in X, Y, Z. Then from (4.5) we get

$$g(K_X Y, Z) = C(X,Y,Z) = \nabla_X g(Y,Z)$$

Hence the difference tensor is same as the skewness operator.

Now let $(\mathbf{M}, g, C)$ be a semi-Riemannian manifold with the trilinear form C and the skewness operator K. Define a linear connection ∇ by $\nabla = \overset{g}{\nabla} - \frac{K}{2}$, where $\overset{g}{\nabla}$ is the Levi-Civita connection. Then, ∇ is torsion free and $\nabla g = C$. Hence $(\mathbf{M}, \nabla, g)$ becomes a statistical manifold.

Definition 4.9. Let $(\mathbf{M}, \nabla, g)$ be a statistical manifold with skewness operator K. The horizontal lift of K, denoted by K^h, is defined as

$$\begin{aligned}K^h(X^h)Y^h &= (K(X)Y)^h, \quad K^h(X^v)Y^v = 0, \\ K^h(X^h)Y^v &= K^h(X^v)Y^h = (K(X)Y)^v,\end{aligned}$$

for $X, Y \in \mathcal{X}(\mathbf{M})$.

Theorem 4.1. *[23] Let $(\mathbf{M}, \nabla, g)$ be a statistical manifold with skewness operator K. Define a linear connection $\hat{\nabla}$ on $T\mathbf{M}$ by*

$$\tilde{\nabla} = \overset{g^s}{\nabla} - \frac{1}{2}K^h,$$

where $\overset{g^s}{\nabla}$ *is the Levi-Civita connection of the Sasaki lift metric* g^s*. Then, the triplet* $(T\mathbf{M}, \tilde{\nabla}, g^s)$ *is a statistical manifold.*

Proof. Consider the cubic form $\tilde{C}$ defined by

$$\tilde{C}(\tilde{X}, \tilde{Y}, \tilde{Z}) = g^s(K^h(\tilde{X})\tilde{Y}, \tilde{Z}), \quad \text{for} \quad \tilde{X}, \tilde{Y}, \tilde{Z} \in \mathcal{X}(T\mathbf{M}).$$

Then by (4.4) we get

$$\tilde{C}(\tilde{X}, \tilde{Y}, \tilde{Z}) = g(\pi_*(K^h(\tilde{X})\tilde{Y}), \pi_*\tilde{Z}) + g(\mathcal{K}(K^h(\tilde{X})\tilde{Y}), \mathcal{K}(\tilde{Z})).$$

Hence by definition of K^h

$$\tilde{C}(\tilde{X}, \tilde{Y}, \tilde{Z}) = g(K(\pi_*\tilde{X})\pi_*\tilde{Y}, \pi_*\tilde{Z}) + 2g(K(\mathcal{K}\tilde{X})\mathcal{K}\tilde{Y}, \mathcal{K}\tilde{Z}).$$

Since $(\mathbf{M}, \nabla, g)$ is a statistical manifold, $g(K(\pi_*\tilde{X})\pi_*\tilde{Y}, \pi_*\tilde{Z})$ and $g(K(\mathcal{K}\tilde{X})\mathcal{K}\tilde{Y}, \mathcal{K}\tilde{Z})$ are symmetric. This implies $\tilde{C}(\tilde{X}, \tilde{Y}, \tilde{Z})$ is totally symmetric, hence $(T\mathbf{M}, \tilde{\nabla}, g^s)$ is a statistical manifold. □

Proposition 4.3. *[23] Let* $(\mathbf{M}, \nabla, g)$ *be a statistical manifold. Then* $(T\mathbf{M}, C^h, g^s)$ *is a statistical manifold, where* C^h *is the horizontal lift of the cubic form* C *of* $(\mathbf{M}, \nabla, g)$*.*

Proof. By definition of horizontal lift $(C(X, Y, Z))^h = C^h(X^h, Y^h, Z^h)$. Since C is symmetric, C^h is also symmetric, hence $(T\mathbf{M}, C^h, g^s)$ is a statistical manifold. □

Remark 4.5. Note that the statistical structure on $T\mathbf{M}$ obtained in theorem (4.1) is different form the statistical structure on $T\mathbf{M}$ obtained in proposition (4.3), in fact the cubic form $\tilde{C}$ in the proof of the theorem (4.1) is different from C^h.

Theorem 4.2. *[23] Let* $(\mathbf{M}, \nabla, g)$ *be a statistical manifold. Then,* $(T\mathbf{M}, K^h, g^h)$ *is a statistical manifold.*

Proof. Let $\tilde{C}$ be the corresponding cubic form of K^h with respect to g^h. That is,

$$\tilde{C}(\tilde{X}, \tilde{Y}, \tilde{Z}) = g^h(K^h(\tilde{X})\tilde{Y}, \tilde{Z}), \quad \text{for} \quad \tilde{X}, \tilde{Y}, \tilde{Z} \in \mathcal{X}(T\mathbf{M}).$$

Now from equation (4.3) we have

$$\tilde{C}(\tilde{X}, \tilde{Y}, \tilde{Z}) = \{C(\pi_*\tilde{X}, \pi_*\tilde{Y}, \mathcal{K}\tilde{Z}) + C(\pi_*\tilde{X}, \mathcal{K}(\tilde{Y}), \pi_*\tilde{Z}) + C(\mathcal{K}\tilde{X}, \pi_*(\tilde{Y}), \pi_*\tilde{Z})\}^v, \quad (4.6)$$

where C is the cubic form with respect to (∇, g). Since $(\mathbf{M}, \nabla, g)$ is a statistical manifold C is totally symmetric, then from (4.6) $\tilde{C}$ is totally symmetric. Hence $(T\mathbf{M}, K^h, g^h)$. $\square$

Remark 4.6. Since the horizontal lift C^h of C is symmetric, the tangent bundle $(T\mathbf{M}, C^h, g^h)$ is also a statistical manifold. Note that this statistical structure is different from the statistical structure obtained in theorem (4.2), in fact the cubic form $\tilde{C}$ is not equal to C^h.

Let $(\mathbf{M}, g)$ be an n-dimensional Riemannian manifold, $\pi : T\mathbf{M} \longrightarrow \mathbf{M}$ be the natural projection. Consider a local coordinate system $(U; x^1, ..., x^n)$ on $\mathbf{M}$, then the induced coordinate system on $\pi^{-1}(U)$ is $(x^1, ..x^n; u^1, ..u^n)$. If $X = X^i \frac{\partial}{\partial x_i}$, then

$$X^v = (X^i)^v \frac{\partial}{\partial u^i}, \quad X^h = (X^i)^v \frac{\partial}{\partial x^i} - (X^j)^v u^k \overset{g}{\Gamma}{}^i_{j,k} \frac{\partial}{\partial u^i},$$

where $\overset{g}{\Gamma}{}^i_{j,k}$ are the Christoffel symbols of the Levi-Civita connection $\overset{g}{\nabla}$. Also, we have

$$[X^v, Y^v] = 0, \quad [X^h, Y^v] = (\overset{g}{\nabla}_X Y)^v, \quad [X^h, Y^h] = [X, Y]^h - (\overset{g}{R}(X, Y)Z)^v, \tag{4.7}$$

for $X, Y, Z \in \mathcal{X}(\mathbf{M})$. Here $\overset{g}{R}$ denotes the Riemannian curvature of $\overset{g}{\nabla}$.

Let $\{\frac{\partial}{\partial x^i} \mid_{(x;u)}, \frac{\partial}{\partial u^i} \mid_{(x;u)}\}$, for $i = 1, 2, \cdots, n$, be a basis for $T_{((x;u))}(T\mathbf{M})$, then the horizontal subspace $\mathcal{H}_{((x;u))}$ is spanned by $\{\frac{\delta}{\delta x^i} \mid_{(x;u)}\}$, where

$$\frac{\delta}{\delta x^i} \mid_{(x;u)} = \frac{\partial}{\partial x^i} \mid_{(x;u)} - u^k \overset{g}{\Gamma}{}^i_{j,k} \frac{\partial}{\partial u^i} \mid_{(x;u)} .$$

To simplify the notations, we use ∂_i, δ_i and $\partial_{\bar{i}}$ instead of $\frac{\partial}{\partial x^i}, \frac{\delta}{\delta x^i}$ and $\frac{\partial}{\partial u^i}$, respectively. Now, represent the equations in (4.7) using local coordinates as

$$[\partial_{\bar{i}}, \partial_{\bar{j}}] = 0, \quad [\delta_i, \partial_{\bar{j}}] = \overset{g}{\Gamma}{}^i_{j,k} \partial_k, \quad [\delta_i, \delta_j] = -u^r \overset{g}{R}{}^k_{ijr} \partial_k.$$

Also, represent the Sasaki lift metric g^s as

$$g^s(\delta_i, \delta_j) = g_{ij}, \quad g^s(\delta_i, \partial_{\bar{i}}) = 0, \quad g^s(\partial_{\bar{i}}, \partial_{\bar{j}}) = g_{ij}.$$

Let $\tilde{\nabla}$ be any torsion-free affine connection on $T\mathbf{M}$. Then, with respect to $\{\delta_i, \partial_{\bar{i}}\}$ we have

$$\begin{aligned}
\tilde{\nabla}_{\delta_i} \delta_j &= \tilde{\Gamma}^k_{ij} \delta_k + \tilde{\Gamma}^{\bar{k}}_{ij} \partial_{\bar{k}}, \quad \tilde{\nabla}_{\delta_i} \partial_{\bar{j}} = \tilde{\Gamma}^k_{i\bar{j}} \delta_k + \tilde{\Gamma}^{\bar{k}}_{i\bar{j}} \partial_{\bar{k}}, \\
\tilde{\nabla}_{\partial_{\bar{i}}} \delta_j &= \tilde{\Gamma}^k_{\bar{i}j} \delta_k + \tilde{\Gamma}^{\bar{k}}_{\bar{i}j} \partial_{\bar{k}}, \quad \tilde{\nabla}_{\partial_{\bar{i}}} \partial_{\bar{j}} = \tilde{\Gamma}^k_{\bar{i}\bar{j}} \delta_k + \tilde{\Gamma}^{\bar{k}}_{\bar{i}\bar{j}} \partial_{\bar{k}},
\end{aligned}$$

where $\tilde{\Gamma}^c_{ab}$, $a, b, c \in \{1, \cdots, n, \overline{1}, \cdots, \overline{n}\}$ are smooth functions on $T\mathrm{M}$.

Lemma 4.1. *[24] The symmetry of $\tilde{\nabla}$ has the following local alternate form:*

i) $\tilde{\Gamma}^k_{i\overline{j}} = \tilde{\Gamma}^k_{\overline{j}i}, \quad \tilde{\Gamma}^{\overline{k}}_{i\overline{j}} - \tilde{\Gamma}^{\overline{k}}_{\overline{j}i} = \overset{g}{\Gamma}{}^k_{i,j}.$

ii) $\tilde{\Gamma}^k_{ij} = \tilde{\Gamma}^k_{ji}, \quad \tilde{\Gamma}^{\overline{k}}_{ij} - \tilde{\Gamma}^{\overline{k}}_{ji} = -u^r R^k_{ijr}.$

iii) $\tilde{\Gamma}^k_{\overline{i}\overline{j}} = \tilde{\Gamma}^k_{\overline{j}\overline{i}}, \quad \tilde{\Gamma}^{\overline{k}}_{\overline{i}\overline{j}} = \tilde{\Gamma}^{\overline{k}}_{\overline{j}\overline{i}}.$

Proof. Since $\tilde{\nabla}$ is torsion-free

$$\tilde{\nabla}_{\delta_j}\partial_{\overline{i}} - \tilde{\nabla}_{\partial_{\overline{i}}}\delta_j = [\delta_j, \partial_{\overline{i}}] = \overset{g}{\Gamma}{}^k_{i,j}\partial_{\overline{k}}.$$

Then, from the above local representation of $\tilde{\nabla}$ we get (i). Similarly, we can prove (ii) and (iii). □

Lemma 4.2. *[24] Let (M, ∇, g) be a statistical manifold. Then, applying the Codazzi equations for $(T\mathrm{M}, \tilde{\nabla}, g^s)$*

i) $\partial_i g_{jk} - \tilde{\Gamma}^r_{ij}g_{rk} - \tilde{\Gamma}^r_{ik}g_{rj} = \partial_j g_{ki} - \tilde{\Gamma}^r_{jk}g_{ri} - \tilde{\Gamma}^r_{ji}g_{rk} = \partial_k g_{ij} - \tilde{\Gamma}^r_{ki}g_{rj} - \tilde{\Gamma}^r_{kj}g_{ri},$

ii) $\tilde{\Gamma}^{\overline{r}}_{ij}g_{rk} + \tilde{\Gamma}^r_{i\overline{k}}g_{jr} = \tilde{\Gamma}^r_{j\overline{k}}g_{ri} + \tilde{\Gamma}^{\overline{r}}_{ji}g_{kr} = \tilde{\Gamma}^r_{\overline{k}i}g_{rj} + \tilde{\Gamma}^r_{\overline{k}j}g_{ri},$

iii) $\partial_i g_{jk} - \tilde{\Gamma}^{\overline{r}}_{i\overline{j}}g_{rk} - \tilde{\Gamma}^{\overline{r}}_{i\overline{k}}g_{jr} = -\tilde{\Gamma}^r_{\overline{j}\overline{k}}g_{ri} - \tilde{\Gamma}^{\overline{r}}_{\overline{j}i}g_{rk} = -\tilde{\Gamma}^{\overline{r}}_{\overline{k}i}g_{rj} - \tilde{\Gamma}^r_{\overline{k}\overline{j}}g_{ir},$

iv) $\tilde{\Gamma}^{\overline{r}}_{\overline{i}\overline{j}}g_{rk} + \tilde{\Gamma}^{\overline{r}}_{\overline{i}k}g_{rj} = \tilde{\Gamma}^{\overline{r}}_{\overline{j}k}g_{ri} + \tilde{\Gamma}^{\overline{r}}_{\overline{j}i}g_{rk} = \tilde{\Gamma}^{\overline{r}}_{k\overline{i}}g_{rj} + \tilde{\Gamma}^{\overline{r}}_{k\overline{j}}g_{ri}.$

Proof. Consider

$$(\tilde{\nabla}_{\delta_i}g^s)(\delta_j, \delta_k) = \partial_i g_{jk} - \tilde{\Gamma}^r_{ij}g_{rk} - \tilde{\Gamma}^r_{ik}g_{rj}.$$

Then by the Codazzi equation

$$(\tilde{\nabla}_{\delta_i}g^s)(\delta_j, \delta_k) = (\tilde{\nabla}_{\delta_j}g^s)(\delta_k, \delta_i) = (\tilde{\nabla}_{\delta_k}g^s)(\delta_i, \delta_j),$$

we get (i). Also, the equations

$$\begin{aligned}
(\tilde{\nabla}_{\delta_i}g^s)(\delta_j, \partial_{\overline{k}}) &= -\tilde{\Gamma}^{\overline{r}}_{ij}g_{rk} - \tilde{\Gamma}^r_{i\overline{k}}g_{jr} \\
(\tilde{\nabla}_{\delta_j}g^s)(\partial_{\overline{k}}, \delta_i) &= -\tilde{\Gamma}^r_{j\overline{k}}g_{ri} - \tilde{\Gamma}^{\overline{r}}_{ji}g_{kr} \\
(\tilde{\nabla}_{\partial_{\overline{k}}}g^s)(\delta_i, \delta_j) &= -\tilde{\Gamma}^r_{\overline{k}i}g_{rj} - \tilde{\Gamma}^r_{\overline{k}j}g_{ri}
\end{aligned}$$

and the Codazzi equation

$$(\tilde{\nabla}_{\delta_i} g^s)(\delta_j, \partial_{\overline{k}}) = (\tilde{\nabla}_{\delta_j} g^s)(\partial_{\overline{k}}, \delta_i) = (\tilde{\nabla}_{\partial_{\overline{k}}} g^s)(\delta_i, \delta_j),$$

imply (ii). Similarly, (iii) and (iv) can be computed. $\square$

Proposition 4.4. *[24] Let (M, ∇, g) be a statistical manifold. Then, $(T\mathrm{M}, \tilde{\nabla}, g^s)$ is a statistical manifold if and only if*

$$\begin{aligned}
(\tilde{\Gamma}^r_{ik} - \Gamma^r_{ik}) g_{rj} &= (\tilde{\Gamma}^r_{jk} - \Gamma^r_{jk}) g_{ri}, && (4.8)\\
\tilde{\Gamma}^{\overline{r}}_{ij} g_{rk} &= \tilde{\Gamma}^r_{\overline{k}j} g_{ri}, && (4.9)\\
\tilde{\Gamma}^r_{i\overline{k}} g_{jr} - u^m \overset{g}{R}_{ijmk} &= \tilde{\Gamma}^r_{j\overline{k}} g_{ri}, && (4.10)\\
(\tilde{\Gamma}^{\overline{r}}_{i\overline{k}} - \overset{g}{\Gamma}{}^r_{ik}) g_{jr} &= \tilde{\Gamma}^r_{\overline{j}\overline{k}} g_{ri}, && (4.11)\\
\tilde{\Gamma}^{\overline{r}}_{\overline{j}i} g_{rk} &= \tilde{\Gamma}^{\overline{r}}_{\overline{k}i} g_{rj}, && (4.12)\\
\tilde{\Gamma}^{\overline{r}}_{\overline{i}\overline{k}} g_{rj} &= \overline{\Gamma}^{\overline{r}}_{\overline{j}\overline{k}} g_{ri}, && (4.13)
\end{aligned}$$

where $\overset{g}{R}_{ijmk} = \overset{g}{R}{}^r_{ijm} g_{rk}$.

Proof. Since (M, ∇, g) is a statistical manifold, from Codazzi equation of (∇, g)

$$\partial_i g_{jk} - \partial_j g_{ki} = \Gamma^r_{ik} g_{rj} - \Gamma^r_{jk} g_{ri}. \tag{4.14}$$

Also from (i) of lemma (4.2),

$$\partial_i g_{jk} - \partial_j g_{ki} = \tilde{\Gamma}^r_{ik} g_{rj} - \tilde{\Gamma}^r_{jk} g_{ri}. \tag{4.15}$$

Now from (4.14) and (4.15) we get (4.8). Also from the first equality of (ii) in lemma (4.2) and from second equality of (ii) of lemma (4.1), we get (4.10). From (ii) of lemma (4.2)

$$\tilde{\Gamma}^{\overline{r}}_{ij} g_{rk} + \tilde{\Gamma}^r_{i\overline{k}} g_{jr} = \tilde{\Gamma}^r_{\overline{k}i} g_{rj} + \tilde{\Gamma}^r_{\overline{k}j} g_{ri} \tag{4.16}$$

and from (i) of lemma(4.1),

$$\tilde{\Gamma}^k_{i\overline{j}} = \tilde{\Gamma}^k_{\overline{j}i}. \tag{4.17}$$

Then, we get (4.9) from (4.16) and (4.17). Now from the second equation of (i) in lemma (4.1) and first equality of (iii) in lemma (4.2),

$$\partial_i g_{jk} - \overset{g}{\Gamma}{}^r_{ij} g_{rk} = \tilde{\Gamma}^{\overline{r}}_{i\overline{k}} g_{jr} - \tilde{\Gamma}^r_{\overline{j}\overline{k}} g_{ri}. \tag{4.18}$$

Since $\overset{g}{\nabla}$ is the Levi-Civita connection of the metric g,

$$\partial_i g_{jk} = \overset{g}{\Gamma}{}^r_{ij} g_{rk} + \overset{g}{\Gamma}{}^r_{ik} g_{jr}. \tag{4.19}$$

From (4.18) and (4.19) we get (4.11). Also, (4.12) is obtained from the last equation (iii) of lemma (4.2) and form first equation in (iii) of lemma (4.1). From (iv) of lemma (4.2) and from the second equation of (ii) of lemma (4.1) we get (4.13). $\square$

Remark 4.7. We discussed various results regarding the statistical manifold structure on $T\mathbf{M}$. Matsuzoe and Inoguchi [23], have shown that $(T\mathbf{M}, \nabla^c, g^c)$, $(T\mathbf{M}, \nabla^h, g^s)$, $(T\mathbf{M}, \nabla^h, g^h)$, $(T\mathbf{M}, \tilde{\nabla}, g^s)$, $(T\mathbf{M}, C^h, g^s)$ and $(T\mathbf{M}, K^h, g^h)$ are statistical manifolds. In [24], Balan et al. have proved that $(T\mathbf{M}, \tilde{\nabla}, g^s)$ is a statistical manifold. In the next section, we give a necessary and sufficient condition for $(T\mathbf{M}, \nabla^c, g^s)$ to become a statistical manifold.

4.1.2 Affine Submersion with Horizontal Distribution and Statistical Manifold Structure on the Tangent Bundle

In this subsection, we obtained a necessary and sufficient condition for $T\mathbf{M}$ to become a statistical manifold with respect to the Sasaki lift metric and the complete lift connection, using affine submersion with horizontal distribution [22].

Consider the submersion $\pi : T\mathbf{M} \longrightarrow \mathbf{M}$. Let ∇ be an affine connection on $\mathbf{M}$. Then, there is a horizontal distribution $\mathcal{H}$ such that

$$T_{(x;u)}(T\mathbf{M}) = \mathcal{H}_{(x;u)}(T\mathbf{M}) \oplus \mathcal{V}_{(x;u)}(T\mathbf{M}).$$

for every $(x; u) \in T\mathbf{M}$.
Now, we show that the submersion π of $T\mathbf{M}$ into $\mathbf{M}$ with the complete lift of affine connection is an affine submersion with horizontal distribution.

Proposition 4.5. *The submersion $\pi : (T\mathbf{M}, \nabla^c) \longrightarrow (\mathbf{M}, \nabla)$ is an affine submersion with horizontal distribution.*

Proof. We need to show that

$$\mathcal{H}(\nabla^c_{X^h} Y^h) = (\nabla_X Y)^h.$$

We have $X^h = X^c - \gamma(\nabla X)$, then

$$\begin{aligned}
\nabla^c_{X^h} Y^h &= \nabla^c_{X^c-\gamma(\nabla X)} Y^c - \gamma(\nabla Y), \\
&= \nabla^c_{X^c-\gamma(\nabla X)} Y^c - \nabla^c_{X^c-\gamma(\nabla X)} \gamma(\nabla Y), \\
&= \nabla^c_{X^c} Y^c - \nabla^c_{\gamma(\nabla X)} Y^c - \nabla^c_{X^c} \gamma(\nabla Y) + \nabla^c_{\gamma(\nabla X)} \gamma(\nabla Y).
\end{aligned}$$

Using $\nabla^c_{X^v} Y^v = 0$ ([23]),

$$\nabla^c_{X^h} Y^h = (\nabla_X Y)^c - [\nabla^c_{\gamma(\nabla X)} Y^c + \nabla^c_{X^c} \gamma(\nabla Y)]. \tag{4.20}$$

By definition

$$(\nabla_X Y)^c = (\nabla_X Y)^h + \gamma(\nabla(\nabla_X Y)). \tag{4.21}$$

From (4.20) and (4.21)

$$\mathcal{H}(\nabla^c_{X^h} Y^h) = (\nabla_X Y)^h.$$

Hence the submersion $\pi : (T\mathbf{M}, \nabla^c) \longrightarrow (\mathbf{M}, \nabla)$ is an affine submersion with horizontal distribution. $\square$

Proposition 4.6. *The submersion $\pi : (T\mathbf{M}, g^s) \longrightarrow (\mathbf{M}, g)$ is a semi-Riemannian submersion.*

Proof. Clearly $\pi^{-1}(p) = T_p\mathbf{M}$ for all $p \in \mathbf{M}$ is a semi-Riemannian submanifold of $T\mathbf{M}$ and by definition of g^s we have

$$g^s(X^h, Y^h) = g(X, Y).$$

Hence π is a semi-Riemannian submersion. $\square$

Now, we give a necessary and sufficient condition for the tangent bundle to be a statistical manifold with the Sasaki lift metric and the complete lift connection.

Theorem 4.3. *$(T\mathbf{M}, \nabla^c, g^s)$ is a statistical manifold if and only if*

1. $\mathcal{H}(S_V X) = A_X V - \overline{A}_X V$, *where* $S_V X = \nabla^c_V X - \overline{\nabla}^c_V X$.
2. $\mathcal{V}(S_X V) = T_V X - \overline{T}_V X$.

3. *$(T_p\mathbf{M}, \hat{\nabla}^c, \hat{g}^s)$ is a statistical manifold for each $p \in \mathbf{M}$, where $\hat{g}^s$ and $\hat{\nabla}^c$ are the induced metric and connection on the fibers.*

4. *$(\mathbf{M}, \nabla, g)$ is a statistical manifold.*

X is a horizontal vector field and V is a vertical vector field.

Proof. From propositions (4.5) and (4.6) we get that $\pi : (T\mathbf{M}, \nabla^c, g^s) \longrightarrow (\mathbf{M}, \nabla, g)$ is an affine submersion with horizontal distribution. Since $g^s(X^H, Y^V) = 0$, take $\mathcal{H}(T\mathbf{M}) = \mathcal{V}(T\mathbf{M})^{\perp}$. First we show that the following equations hold for horizontal vectors X, Y and vertical vectors U, V, W.

$$
\begin{aligned}
(\nabla^c_V g^s)(X,Y) &= -g^s(S_V X, Y), && (4.22)\\
(\nabla^c_X g^s)(V,Y) &= -g^s(A_X V, Y) + g^s(\overline{A}_X V, Y), && (4.23)\\
(\nabla^c_X g^s)(V,W) &= -g^s(S_X V, W), && (4.24)\\
(\nabla^c_V g^s)(X,W) &= -g^s(T_V X, W) + g^s(\overline{T}_V X, W), && (4.25)\\
(\nabla^c_U g^s)(V,W) &= (\hat{\nabla}^c_U \hat{g}^s)(V,W), && (4.26)\\
(\nabla^c_{\tilde{X}} g^s)(\tilde{X}_1, \tilde{X}_2) &= (\nabla_X g)(X_1, X_2), && (4.27)
\end{aligned}
$$

where $\tilde{X}_i$ are the horizontal lift of vector fields X_i on $\mathbf{M}$ and $S_V X = \nabla^c_V X - \overline{\nabla}^c_V X$. To see (4.22) consider

$$
\begin{aligned}
(\nabla^c_V g^s)(X,Y) &= V g^s(X,Y) - g^s(\nabla^c_V X, Y) - g^s(X, \nabla^c_V Y),\\
&= g^s(\overline{\nabla}^c_V X, Y) - g^s(X, \nabla^c_V Y),\\
&= -g^s(S_V X, Y).
\end{aligned}
$$

Similarly, we can prove the other equations. Now, suppose $(T\mathbf{M}, \nabla^c, g^s)$ is a statistical manifold, then $\nabla^c g^s$ is symmetric. From (4.22) and (4.23)

$$\mathcal{H}(S_V X) = A_X V - \overline{A}_X V.$$

From (4.24) and (4.25)

$$\mathcal{V}(S_X V) = T_V X - \overline{T}_V X$$

and from (4.26) $\hat{\nabla}^c \hat{g}^s$ is symmetric, so $(T_p\mathbf{M}, \hat{\nabla}^c, \hat{g}^s)$ is a statistical manifold for each $p \in \mathbf{M}$. Also from (4.27) the triplet $(\mathbf{M}, \nabla, g)$ is a statistical manifold.

Conversely, if all the four conditions hold then from the above equations $\nabla^c g^s$ is symmetric, so $(T\mathbf{M}, \nabla^c, g^s)$ is a statistical manifold. $\square$

4.2 Harmonic Maps Between Statistical Manifolds

The motivation to study harmonic maps comes from the applications of Riemannian submersion in theoretical physics [25]. Harmonic maps formulation of field theories lead to a geometrical description in the unified field theory program. Presently, we see an increasing interest in harmonic maps between statistical manifolds [26], [27]. In [26], Uohashi obtained a condition for the harmonicity on α-conformally equivalent statistical manifolds. In this section, we first look at the definition of the harmonic map using tension field. Then, prove a necessary and sufficient condition for the harmonicity of the identity map for conformally-projectively equivalent statistical manifolds. The conformal statistical submersion is defined which is a generalization of the statistical submersion and proved that harmonicity and conformality cannot coexist [28].

4.2.1 Harmonic Maps

In this subsection, we discuss the basic ideas to define the tension field and then the definition of a harmonic map is given [57].

Let $(\mathbf{M}, g_m)$ and $(\mathbf{B}, g_b)$ be Riemannian manifolds with dimensions n and m, respectively and let $\{x^1, \cdots, x^n\}$ be a local coordinate on $\mathbf{M}$. Then, the Riemannian metric g_m induces the natural linear isomorphisms, $\flat : T_p\mathbf{M} \longrightarrow T_p\mathbf{M}^*$ and $\sharp : T_p\mathbf{M}^* \longrightarrow T_p\mathbf{M}$ defined by

$$\flat(X_p) = X_p^\flat = \sum_{i=1}^{n}\left(\sum_{i=1}^{n}(g_m)_{ij}(p)X^j(p)\right)(dx^i)_p \tag{4.28}$$

and

$$\sharp(\omega_p) = \omega_p^\sharp = \sum_{i=1}^{n}\left(\sum_{i=1}^{n}g_m^{ij}(p)\omega_j(p)\right)(\frac{\partial}{\partial x^i})_p, \tag{4.29}$$

where X, ω denotes the vector field and the 1-from on $\mathbf{M}$, respectively. Note that $\flat$ and $\sharp$ are inverses of each other, that is $T_p\mathbf{M}$ and $T_p\mathbf{M}^*$ are isomorphic. Now define a metric $(g_m^*)_p$ on $T_p\mathbf{M}^*$ by

$$(g_m^*)_p(\omega_p, \theta_p) = (g_m)_p(\omega_p^\sharp, \theta_p^\sharp) \quad \text{for} \quad \omega_p, \theta_p \in T_p\mathbf{M}^*. \tag{4.30}$$

Since $\sharp$ is linear, g_m^* is a metric. Also, $g_m^{ij} = g_m^*(dx^i, dx^i)$, where g_m^{ij} is the inverse of $(g_m)_{ij}$.

Let $\overset{g_m}{\nabla}$ denote the Levi-Civta connection on $\mathbf{M}$, define an affine connection $\overset{g_m}{\nabla}{}^*$ on $T\mathbf{M}^*$ using isomorphisms $\flat$ and $\sharp$, as follows

$$\begin{aligned}\overset{g_m}{\nabla}{}^*_X\omega(Y) &= (\overset{g_m}{\nabla}_X\omega^\sharp)^\flat(Y) \\ &= (g_m)_p(\nabla_X\omega^\sharp, Y) \\ &= X((g_m)_p(\omega^\sharp, Y)) - (g_m)_p(\omega^\sharp, \nabla_X Y) \quad (\text{using } \overset{g_m}{\nabla} g_m = 0) \\ &= X\omega(Y) - \omega(\overset{g_m}{\nabla}_X Y),\end{aligned} \tag{4.31}$$

where X and ω denote the vector field and 1-form on $\mathbf{M}$. This connection $\overset{g_m}{\nabla}{}^*$ is called the dual connection of $\overset{g_m}{\nabla}$.

Let $\phi : \mathbf{M} \longrightarrow \mathbf{B}$ be a smooth map and $\pi : \mathbf{E} \longrightarrow \mathbf{B}$ be a vector bundle. Consider the bundle over $\mathbf{M}$ whose fibre over $p \in \mathbf{M}$ is $\mathbf{E}_{\phi(p)}$, the fibre of $\mathbf{E}$ over $\phi(p)$. This bundle is denoted by $\pi^{-1}(\mathbf{E})$, called the pullback bundle over $\mathbf{M}$. In particular, if $\pi : T\mathbf{B} \longrightarrow \mathbf{B}$ is the tangent bundle, then $\pi^{-1}(T\mathbf{B})$ is a subbundle of $T\mathbf{M}$, whose fibre over $p \in \mathbf{M}$ is $T_{\phi(p)}\mathbf{B}$. Let $\{y^1, \cdots, y^m\}$ be the local coordinates on $\mathbf{B}$, then at each point $p \in \mathbf{M}$

$$\left\{ \big(\frac{\partial}{\partial y^1} \circ \phi\big)(p), \cdots, \big(\frac{\partial}{\partial y^m} \circ \phi\big)(p) \right\} \tag{4.32}$$

form a basis for the fibre $T_{\phi(p)}\mathbf{B}$ of $\phi^{-1}(T\mathbf{B})$ over p.

Definition 4.10. Let $(\mathbf{M}, g_m)$ and $(\mathbf{B}, g_b)$ be two Riemannian manifolds with local coordinates $\{x^1, \cdots, x^n\}$ and $\{y^1, \cdots, y^m\}$, respectively. For a smooth map $\phi : \mathbf{M} \longrightarrow \mathbf{B}$, define an affine connection $\tilde{\nabla}$ on $\phi^{-1}(T\mathbf{B})$ from the Levi-Civita connection $\overset{g_b}{\nabla}$ on $\mathbf{B}$ as follows

$$\tilde{\nabla}_{\frac{\partial}{\partial x^i}}\big(\frac{\partial}{\partial y^\gamma} \circ \phi\big) = \overset{g_b}{\nabla}_{\phi_*(\frac{\partial}{\partial x^i})}\frac{\partial}{\partial y^\gamma}. \tag{4.33}$$

From equations (4.32) and (4.33) we have

$$\tilde{\nabla}_{\frac{\partial}{\partial x^i}}\big(\frac{\partial}{\partial y^\gamma} \circ \phi\big) = \sum_{\alpha=1}^{n}\left(\sum_{\beta=1}^{n} \frac{\partial \phi^\beta}{\partial x^i}(p)\overset{g_b}{\Gamma}{}^\alpha_{\beta\gamma}(\phi(p))\right)\big(\frac{\partial}{\partial y^\alpha} \circ \phi\big), \tag{4.34}$$

where $\overset{g_b}{\Gamma}{}^\alpha_{\beta\gamma}$ are the Christoffel symbols of the Levi-Civita connection $\overset{g_b}{\nabla}$ on $\mathbf{B}$.

Now, define an affine connection ∇ on the manifold $T\mathbf{M}^* \otimes \phi^{-1}(T\mathbf{B})$ using $\overset{g_m}{\nabla}{}^*$ and $\tilde{\nabla}$ as follows

$$\nabla_X(\omega \otimes W) = (\overset{g_m}{\nabla}{}^*\omega) \otimes W + \omega \otimes (\tilde{\nabla}_X W), \tag{4.35}$$

where ω is a 1-form on $\mathbf{M}$, $W \in \mathcal{X}(\phi^{-1}(T\mathbf{B}))$ and $X \in \mathcal{X}(\mathbf{M})$. The linearity property can be verified, also

$$\begin{aligned}
\nabla_X(f\omega \otimes W) &= (\overset{gm}{\nabla}{}^* f\omega) \otimes W + f\omega \otimes (\tilde{\nabla}_X W) &(4.36)\\
&= \left(X(f)\omega + f\overset{gm}{\nabla}{}^*_X\omega\right) \otimes W + f\omega \otimes (\tilde{\nabla}_X W) &(4.37)\\
&= X(f)(\omega \otimes W) + f\nabla_X(\omega \otimes W), &(4.38)
\end{aligned}$$

proves that ∇ is in fact a connection in the tensor product $T\mathbf{M}^* \otimes \phi^{-1}(T\mathbf{B})$.

Consider the smooth map $\phi : \mathbf{M} \longrightarrow \mathbf{B}$, then $d\phi_p (= \phi_{*p})$ defines a linear map between $T_p\mathbf{M}$ onto $T_{\phi(p)}\mathbf{B}$, Since $\mathrm{Hom}(T_p\mathbf{M}, T_{\phi(p)}\mathbf{B})$ is isomorphic to $T_p\mathbf{M}^* \otimes T_{\phi(p)}\mathbf{B}$ we can consider $d\phi$ as a smooth section on the vector bundle $T\mathbf{M}^* \otimes \phi^{-1}(T\mathbf{B})$. Here $\mathrm{Hom}(T_p\mathbf{M}, T_{\phi(p)}\mathbf{B})$ denotes the set of all linear maps between $T_p\mathbf{M}$ and $T_{\phi(p)}\mathbf{B}$.

Definition 4.11. Let $\phi : \mathbf{M} \longrightarrow \mathbf{B}$ be a smooth map, then the second fundamental form of ϕ is defined as the covariant derivative of $d\phi$ with respect to the affine connection ∇ on $T\mathbf{M}^* \otimes \phi^{-1}(T\mathbf{B})$. That is, $\nabla d\phi$ is called the second fundamental form of ϕ.

Lemma 4.3. *Let* $\phi : \mathbf{M} \longrightarrow \mathbf{B}$ *be a smooth map and* $X, Y \in \mathcal{X}(\mathbf{M})$. *Then,*

$$\nabla d\phi(X, Y) = \tilde{\nabla}_X \phi_* Y - \phi_*(\overset{gm}{\nabla}_X Y), \tag{4.39}$$

where ∇ *is the affine connection on* $T\mathbf{M}^* \otimes \phi^{-1}(T\mathbf{B})$.

Proof. Consider

$$\begin{aligned}
\big(\nabla(\omega \otimes W)\big)(X, Y) &= \big(\overset{gm}{\nabla}{}^*_X\omega \otimes W + \omega \otimes \tilde{\nabla}_X W\big)(Y)\\
&= \big(X\omega(Y) - \omega(\overset{gm}{\nabla}_X Y)\big) \otimes W + \omega(Y) \otimes \tilde{\nabla}_X W \quad \text{by (4.31)}\\
&= \tilde{\nabla}_X(\omega \otimes W)(Y) - (\omega \otimes W)(\overset{gm}{\nabla}_X Y),
\end{aligned}$$

where ω is a 1-form on $\mathbf{M}$, $W \in \mathcal{X}(\phi^{-1}(T\mathbf{B}))$ and $X \in \mathcal{X}(\mathbf{M})$. Then,

$$\nabla d\phi(X, Y) = \tilde{\nabla}_X d\phi(Y) - d\phi(\overset{g}{\nabla}_X Y).$$

That is,

$$\nabla d\phi(X, Y) = \tilde{\nabla}_X \phi_* Y - \phi_*(\overset{gm}{\nabla}_X Y).$$

Hence proved. This may be regarded as another definition of the second fundamental form. $\square$

Let $\{e_1, \cdots, e_n\}$ be a local orthonormal frame of the tangent space $T_p\mathbf{M}$, $p \in \mathbf{M}$. Trace of the second fundamental form $\nabla d\phi$ of a smooth map $\phi : \mathbf{M} \longrightarrow \mathbf{B}$ is defined as

$$\mathrm{trace}\nabla d\phi(p) = \sum_{i=1}^{n} \nabla d\phi(p)(e_i, e_i). \tag{4.40}$$

The tensor field $\tau(\phi) = \mathrm{trace}\nabla d\phi$ on $\phi^{-1}T\mathbf{B}$ is called the tension field of ϕ.

Definition 4.12. . Let $(\mathbf{M}, g_m)$ and $(\mathbf{B}, g_b)$ be two Riemannian manifolds of dimension n and m, respectively. A smooth map $\phi : \mathbf{M} \longrightarrow \mathbf{B}$ is said to be a harmonic map if its tension field $\tau(\phi)$ is identically zero; namely,

$$\tau(\phi) = \mathrm{trace}\nabla d\phi \equiv 0 \tag{4.41}$$

holds in $\mathbf{M}$. Equation(4.41) is called the equation of harmonic maps.

Now, we look at the coordinate representation for the equation of harmonic maps. Let $\{x^1, \cdots, x^n\}$ and $\{y^1, \cdots, y^m\}$ be the local coordinate systems in $\mathbf{M}$ and $\mathbf{B}$, respectively. With this coordinate, express ϕ as

$$\phi(x) = \big(\phi^1(x^1, \cdots, x^n), \cdots, \phi^m(x^1, \cdots, x^n)\big)$$

and the tension field $\tau(\phi)$ of ϕ as

$$\tau(\phi) = \sum_{a=1}^{m} \tau(\phi)^a \frac{\partial}{\partial y^a} \circ \phi, \tag{4.42}$$

where

$$\begin{aligned} \tau(\phi)^a &= \sum_{i,j=1}^{n} g_m^{ij} \Big\{ \frac{\partial^2 \phi^a}{\partial x^i \partial x^j} - \sum_{k=1}^{n} \overset{g_m}{\Gamma}{}^k_{ij} \frac{\partial \phi^a}{\partial x^k} + \sum_{b,c=1}^{m} \overset{g_b}{\Gamma}{}^a_{b,c}(\phi) \frac{\partial \phi^b}{\partial x^i} \frac{\partial \phi^c}{\partial x^j} \Big\} \\ &= \Delta\phi^a + \sum_{i,j=1}^{n} \sum_{b,c=1}^{m} g_m^{ij} \overset{g_b}{\Gamma}{}^a_{b,c}(\phi) \frac{\partial \phi^b}{\partial x^i} \frac{\partial \phi^c}{\partial x^j}, \end{aligned} \tag{4.43}$$

where $\overset{g_m}{\Gamma}{}^k_{ij}$ and $\overset{g_b}{\Gamma}{}^a_{b,c}$ represents the connection coefficients of the Levi-Civita connection in $\mathbf{M}$ and $\mathbf{B}$, respectively. Therefore, from (4.41) and (4.43) we get the coordinate equation of the harmonic maps as

$$\Delta\phi^a + \sum_{i,j=1}^{n}\sum_{b,c=1}^{m} g_m^{ij}\,{}^{g_b}\Gamma^a_{b,c}(\phi)\frac{\partial\phi^b}{\partial x^i}\frac{\partial\phi^c}{\partial x^j} = 0, \quad 1 \le a \le m.$$

4.2.2 Harmonic Map between Statistical Manifolds

In this subsection, we first discuss Uohashi's result on harmonicity of the identity map for α-conformally equivalent statistical manifolds [26]. We obtained a necessary and sufficient condition for the harmonicity of the identity map for conformally-projectively equivalent statistical manifolds. Then, defined the conformal statistical submersion which is a generalization of the statistical submersion and proved that harmonicity and conformality cannot coexist [28].

Let $(\mathbf{M}, \nabla, g_m)$ and $(\mathbf{B}, \nabla', g_b)$ be two statistical manifolds of dimensions n and m, respectively. Let $\{x^1, x^2, ...x^n\}$ be a local coordinate system on $\mathbf{M}$. We set $(g_m)_{ij} = g_m(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$ and $[g_m^{ij}] = [(g_m)_{ij}]^{-1}$.

Definition 4.13. A smooth map $\pi : (\mathbf{M}, \nabla, g_m) \longrightarrow (\mathbf{B}, \nabla', g_b)$ is said to be a harmonic map relative to (g_m, ∇, ∇') if the tension field $\tau_{(g_m,\nabla,\nabla')}(\pi)$ of π vanishes at each point $p \in \mathbf{M}$, where $\tau_{(g_m,\nabla,\nabla')}(\pi)$ is defined as

$$\tau_{(g_m,\nabla,\nabla')}(\pi) = \sum_{i,j=1}^{n} g_m^{ij}\left\{\hat{\nabla}_{\frac{\partial}{\partial x^i}}\left(\pi_*(\frac{\partial}{\partial x^j})\right) - \pi_*\left(\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}\right)\right\}, \tag{4.44}$$

where $\hat{\nabla}$ is the pullback of the connection ∇' of $\mathbf{B}$ to the induced vector bundle $\pi^{-1}(T\mathbf{B}) \subset T(\mathbf{M})$ and $\hat{\nabla}_{\frac{\partial}{\partial x^i}}\left(\pi_*(\frac{\partial}{\partial x^j})\right) = \nabla'_{\pi_*(\frac{\partial}{\partial x^i})}\pi_*(\frac{\partial}{\partial x^j})$.

Note. Note that we are considering the general affine connections here, not necessarily the Levi-Civita connections as in the Riemannian manifold theory.

Let $(\mathbf{M}, \nabla, g_m)$ and $(\mathbf{M}, \tilde{\nabla}, \tilde{g}_m)$ be statistical manifolds. Then, the identity map $id : (\mathbf{M}, \nabla, g_m) \longrightarrow (\mathbf{M}, \tilde{\nabla}, \tilde{g}_m)$ is a harmonic map relative to $(g_m, \nabla, \tilde{\nabla})$ if

$$\tau_{(g_m,\nabla,\tilde{\nabla})}(id) \;=\; \sum_{i,j=1}^{n} g_m^{ij}(\tilde{\nabla}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j} - \nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}) \tag{4.45}$$

vanishes identically on $\mathbf{M}$. For α-conformally equivalent statistical manifolds Uohashi proved the following.

Proposition 4.7. *[26] Let* $(\mathbf{M}, \nabla, g_m)$ *and* $(\mathbf{M}, \tilde{\nabla}, \tilde{g}_m)$ *be* α*-conformally equivalent statistical manifolds of dimension* $n \geq 2$*. If* $\alpha = -\frac{(n-2)}{(n+2)}$ *or* ϕ *is a constant function on* $\mathbf{M}$*, then the identity map* $id : (\mathbf{M}, \nabla, g_m) \longrightarrow (\mathbf{M}, \tilde{\nabla}, \tilde{g}_m)$ *is a harmonic map relative to* $(g_m, \nabla, \tilde{\nabla})$*.*

Proof. By the definition of α-conformally equivalent statistical manifolds and from the equation (4.45), for k = 1,2,..n,

$$
\begin{aligned}
g_m\left(\tau_{(g_m,\nabla,\tilde{\nabla})}(id), \frac{\partial}{\partial x_k}\right) &= g_m\left(\sum_{i,j=1}^{n} g_m^{ij}\left(\tilde{\nabla}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j} - \nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}\right), \frac{\partial}{\partial x_k}\right) \\
&= \sum_{i,j=1}^{n} g_m^{ij}\Bigg\{ -\frac{1+\alpha}{2} d\phi\left(\frac{\partial}{\partial x^k}\right) g_m\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \\
&+ \frac{1-\alpha}{2}\left\{ d\phi\left(\frac{\partial}{\partial x^i}\right) g_m\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right) + d\phi\left(\frac{\partial}{\partial x^j}\right) g_m\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^k}\right)\right\}\Bigg\} \\
&= \sum_{i,j=1}^{n} g_m^{ij}\left\{ -\frac{1+\alpha}{2}\frac{\partial\phi}{\partial x^k}(g_m)_{ij} + \frac{1-\alpha}{2}\left(\frac{\partial\phi}{\partial x^i}(g_m)_{jk} + \frac{\partial\phi}{\partial x^j}(g_m)_{ik}\right)\right\} \\
&= \left(-\frac{1+\alpha}{2}n + \frac{1-\alpha}{2}2\right)\frac{\partial\phi}{\partial x^k} \\
&= -\frac{1}{2}\left((n+2)\alpha + (n-2)\right)\frac{\partial\phi}{\partial x^k}.
\end{aligned}
$$

Therefore, if $\tau_{(g,\nabla,\tilde{\nabla})}(id) = 0$, it holds that $\alpha = -\frac{(n-2)}{(n+2)}$ or ϕ is a constant function on $\mathbf{M}$. Hence proved. ☐

Now, we have the following theorem.

Theorem 4.4. *Let* $(\mathbf{M}, \nabla, g_m)$ *and* $(\mathbf{M}, \tilde{\nabla}, \tilde{g}_m)$ *be conformally-projectively equivalent statistical manifolds of dimension* n*. Then, the identity map* $id : (\mathbf{M}, \nabla, g_m) \longrightarrow (\mathbf{M}, \tilde{\nabla}, \tilde{g}_m)$ *is a harmonic map if and only if* $\phi = \frac{n}{2}\psi + c$*, where* c *is some constant.*

Proof. By the definition of conformally-projectively equivalent statistical manifolds and from the equation (4.45), for k = 1,2,..n

$$
\begin{aligned}
\tau_{(g_m,\nabla,\tilde{\nabla})}(id) &= \sum_{i,j=1}^{n} g_m^{ij}\left(\tilde{\nabla}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j} - \nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}\right) \\
&= \sum_{i,j=1}^{n} g_m^{ij}\left(-g_m(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) grad_{g_m}\psi + d\phi\left(\frac{\partial}{\partial_{x_j}}\right)\frac{\partial}{\partial x_i} + d\phi\left(\frac{\partial}{\partial_{x_i}}\right)\frac{\partial}{\partial x_j}\right) \\
&= \sum_{i,j=1}^{n} g_m^{ij}\left((-g_{ij}) grad_{g_m}\psi + \frac{\partial\phi}{\partial x_j}\frac{\partial}{\partial x_i} + \frac{\partial\phi}{\partial x_i}\frac{\partial}{\partial x_j}\right).
\end{aligned}
$$

Now,

$$
\begin{aligned}
g_m\left(\tau_{(g_m,\nabla,\tilde{\nabla})}(id), \frac{\partial}{\partial x_k}\right) &= -ng_m\left(grad_{g_m}\psi, \frac{\partial}{\partial x_k}\right) + \sum_{i=1}^{k}\frac{\partial\phi}{\partial x_i}\delta_{ik} + \sum_{j=1}^{k}\frac{\partial\phi}{\partial x_j}\delta_{jk} \\
&= -n\frac{\partial\psi}{\partial x_k} + \sum_{i=1}^{k}\frac{\partial\phi}{\partial x_i}\delta_{ik} + \sum_{j=1}^{k}\frac{\partial\phi}{\partial x_j}\delta_{jk} \\
&= -n\frac{\partial\psi}{\partial x_k} + 2\frac{\partial\phi}{\partial x_k}. \qquad (4.46)
\end{aligned}
$$

From the equation (4.46), id is harmonic if and only if $\frac{\partial\phi}{\partial x_k} = \frac{n}{2}\frac{\partial\psi}{\partial x_k}$ for all $k \in \{1, 2, ...n\}$. Hence, id is harmonic if and only if $\phi = \frac{n}{2}\psi + c$, where c is some constant. □

Now, we define the conformal statistical submersion which is a generalization of the statistical submersion.

Definition 4.14. Let $(\mathbf{M}, \nabla, g_m)$ and $(\mathbf{B}, \nabla', g_b)$ be two statistical manifolds of dimensions n and m, respectively $(n \geq m)$. A submersion $\pi : (\mathbf{M}, \nabla, g_m) \longrightarrow (\mathbf{B}, \nabla', g_b)$ is called a conformal statistical submersion if there exists a smooth function ϕ on $\mathbf{M}$ such that

$$
\begin{aligned}
g_m(X, Y) &= e^{2\phi} g_b(\pi_* X, \pi_* Y), \qquad (4.47) \\
\pi_*(\nabla_X Y) &= \nabla'_{\pi_* X}\pi_* Y + X(\phi)\pi_* X + Y(\phi)\pi_* X - \pi_*(grad_\pi \phi) g_m(X, Y), \qquad (4.48)
\end{aligned}
$$

for basic vector fields X and Y on $\mathbf{M}$.

Note. If ϕ is a constant, then π is a statistical submersion. Also, note that conformal statistical submersions are conformal submersions with horizontal distribution.

Next, we prove that harmonicity and conformality cannot coexist.

Theorem 4.5. *Let $\pi : (\mathbf{M}, \nabla, g_m) \longrightarrow (\mathbf{B}, \nabla', g_b)$ be a conformal statistical submersion. Then, π is a harmonic map if and only if ϕ is constant.*

Proof. Assume that ϕ is a constant, then by equations (4.48) and (4.44) we get π is a harmonic map. Conversely, assume π is harmonic. Now, consider the equations

$$
\begin{aligned}
\tau(\pi) &= \sum_{i,j=1}^{n} g_m^{ij}\left\{\tilde{\nabla}_{\frac{\partial}{\partial x^i}}\left(\pi_*(\frac{\partial}{\partial x^j})\right) - \pi_*\left(\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}\right)\right\} \\
&= n\pi_*(grad_\pi\phi) - 2\sum_{i,j=1}^{n}\frac{\partial\phi}{\partial x_i}g_m^{ij}\pi_*(\frac{\partial}{\partial x_j}). \qquad (4.49)
\end{aligned}
$$

and

$$\begin{aligned}
g_b(\tau(\pi), \pi_* \frac{\partial}{\partial x_k}) &= ne^{-2\phi} g_m(grad_\pi \phi, \frac{\partial}{\partial x_k}) - 2e^{-2\phi} \sum_{i,j=1}^{m} \frac{\partial \phi}{\partial x_i} g_m^{ij} g_{mjk} \\
&= ne^{-2\phi} g_m(grad_\pi \phi, \frac{\partial}{\partial x_k}) - 2e^{-2\phi} \left(\sum_{i=1}^{n} \frac{\partial \phi}{\partial x_i} \right) \\
&= ne^{-2\phi} \frac{\partial \phi}{\partial x_k} - 2e^{-2\phi} \left(\sum_{i=1}^{n} \frac{\partial \phi}{\partial x_i} \right). \qquad (4.50)
\end{aligned}$$

Since π is harmonic, from (4.50) we get

$$n \frac{\partial \phi}{\partial x_k} = 2 \sum_{i=1}^{n} \frac{\partial \phi}{\partial x_i}, \qquad (4.51)$$

for each $k \in \{1, 2, ..n\}$. That is, we have the system of equations

$$\begin{bmatrix} 2-n & 2 & 2 & \dots & 2 \\ 2 & 2-n & 2 & \dots & 2 \\ \dots & \dots & \dots & \dots & \dots \\ 2 & 2 & 2 & \dots & 2-n \end{bmatrix} \begin{bmatrix} \frac{\partial \phi}{\partial x_1} \\ \frac{\partial \phi}{\partial x_2} \\ \dots \\ \frac{\partial \phi}{\partial x_n} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dots \\ 0 \end{bmatrix}.$$

Since for each fixed n the above $n \times n$ matrix is invertible we get $\frac{\partial \phi}{\partial x_k} = 0$ for all k. Hence, ϕ is a constant. Thus, π is a statistical submersion. $\square$

4.3 Harmonic Maps between Tangent Bundles of Statistical Manifolds

Harmonicity of the tangent maps of tangent bundles endowed with the Sasaki lift metric were studied in [58], [59] for Riemannian manifolds. In [30], Oproiu obtained conditions for the tangent map to be harmonic in the case of tangent bundles equipped with the metrics obtained from the complete lift of metrics and the vertical lift of appropriate tensor fields. In this section, certain properties of the differential of the tangent map is given first. For statistical manifolds, we proved that a smooth map $\phi : \mathbf{M} \longrightarrow \mathbf{B}$ is harmonic with respect to ∇ and ∇^* if and only if it is harmonic with respect to the conjugate connections $\overline{\nabla}$ and $\overline{\nabla^*}$. Then, given a necessary condition for the harmonicity of the tangent map with respect to the complete lift structure on the tangent bundles. Also, prove $\pi : (\mathbf{M}, \nabla, g_m) \longrightarrow$

$(\mathbf{B}, \nabla^*, g_b)$ is a statistical submersion if and only if $\pi_* : (T\mathbf{M}, \nabla^c, g_m^c) \longrightarrow (T\mathbf{B}, \nabla^{*c}, g_b^c)$ is a statistical submersion.

4.3.1 Harmonic Maps between Tangent bundles

In this subsection, we first discuss certain properties of the differential of the tangent map. Then, the result of Oproiu [30] on harmonicity of the tangent map is discussed.

Let $(\mathbf{M}, g)$ be an n-dimensional Riemannian manifold with the natural projection map $\pi : T\mathbf{M} \longrightarrow \mathbf{M}$. Let $(U; x^1, ..., x^n)$ be the local coordinates on $\mathbf{M}$ with the induced coordinate system $(x^1, ..x^n; u^1, ..u^n)$ on $\pi^{-1}(U)$. Note that the vector fields $\frac{\delta}{\delta x^i} = \left(\frac{\partial}{\partial x^i}\right)^h$, for $i = 1, \cdots, n$ define a local frame in the horizontal distribution $\mathcal{H}(T\mathbf{M})$ on $T\mathbf{M}$ defined by $\overset{g}{\nabla}$ and the vertical fields $\frac{\partial}{\partial u^i} = \left(\frac{\partial}{\partial x^i}\right)^v$, for $i = 1, \cdots, n$ define a local frame in the vertical distribution $\mathcal{V}(T\mathbf{M}) = ker\ \pi_*$. The system of local 1-froms $(dx^i, \delta u^i)$ for $i = 1, \cdots, n$ on $T\mathbf{M}$ defines the local dual frame of the local frame $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial u^i}\right)$, where

$$\delta u^i = du^i + \overset{g}{\Gamma}{}^j_{j,i} u^i dx^j.$$

Let c be a symmetric tensor field of type $(0, 2)$ on $\mathbf{M}$. Then, define a semi-Riemannian metric G on $T\mathbf{M}$ as follows

$$G(X^h, Y^h) = c(X, Y), \quad G(X^h, X^v) = G(Y^v, X^h) = g(X, Y), \quad G(X^v, Y^v) = 0,$$

for $X, Y \in \mathcal{X}(\mathbf{M})$.

Let $g = g_{ij} dx^i dx^j$ and $c = c_{ij} dx^i dx^j$ be the local coordinate expressions of g and c, respectively. Then, G can be represented in local coordinates as

$$G = 2g_{ij} \delta u^i dx^j + c_{ij} dx^i dx^j.$$

Thus, the semi-Riemannian metric $G = g^c + c^v$, where g^c is the complete lift of g and c^v is the vertical lift of c. In [30], Oproiu computed the Levi-Civita connection $\overset{G}{\nabla}$ of the semi-Riemannian metric G as follows.

$$\begin{aligned} \overset{G}{\nabla}_{X^v} Y^v &= 0, \quad \overset{G}{\nabla}_{X^v} Y^h = 0, \quad , \overset{G}{\nabla}_{X^h} Y^v = \left(\overset{g}{\nabla}_X Y\right)^v, \\ \overset{G}{\nabla}_{X^h} Y^h &= \left(\overset{g}{\nabla}_X Y\right)^h + A(X^h, Y^h), \quad \text{for } X, Y \in \mathcal{X}(\mathbf{M}), \end{aligned}$$

where A is a bilinear map from $\mathcal{H}(T\mathbf{M})$ to $\mathcal{V}(T\mathbf{M})$ having the local coordinate expression

$$A(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}) = A^k_{ij}\frac{\partial}{\partial u^k}; \quad A^k_{ij} = -\overset{g}{R}{}^k_{jio} + \frac{1}{2}\left(\overset{g}{\nabla}_i c^k_j + \overset{g}{\nabla} j c^k_i - \overset{g}{\nabla}{}^k c_{ij}\right). \tag{4.52}$$

Here $\overset{g}{R}{}^k_{jio} = \overset{g}{R}{}^k_{jih} u^h$, $\overset{g}{R}{}^k_{jih}$ is the local coordinate expression of the curvature tensor field $\overset{g}{R}$ of $\overset{g}{\nabla}$ and $\overset{g}{\nabla}_i c^k_j$ denotes the covariant derivative of the components of the tensor field c.

Let $(\mathbf{B}, g')$ be a Riemannian manifold with dimension m. Consider a smooth map $\phi : \mathbf{M} \longrightarrow \mathbf{B}$ and $\phi^{-1}(T\mathbf{B})$ be the pullback bundle of the tangent bundle $T\mathbf{B}$ over $\mathbf{M}$ by ϕ. We denote $\tilde{\nabla}$ the pullback of the Levi-Civita connection $\overset{g'}{\nabla}$ on $\mathbf{B}$ (cf. Definition (4.10)). Follow the convention that the indices h, i, j, k, l run over the set $\{1, \cdots, n\}$ and the indices $\alpha, \beta, \gamma, \sigma$ run over the set $\{1, \cdots, m\}$. Let (V, y^α) be a local coordinate on $\mathbf{B}$, for simplicity use the same notation $\frac{\partial}{\partial y^\alpha}$, for $\alpha = 1, \cdots, m$, for the local frames in $T\mathbf{B}$ and the corresponding local frames in $\phi^{-1}(T\mathbf{B})$. Let

$$y^\alpha = \phi^\alpha(x^1, \cdots, x^n), \quad \text{for} \;\; \alpha = 1, \cdots, m$$

be the local coordinate expression of ϕ. Then,

$$\phi_*\left(\frac{\partial}{\partial x^j}\right) = \phi^\alpha_j \frac{\partial}{\partial y^\alpha} \quad \text{where} \quad \phi^\alpha_j = \frac{\partial \phi^\alpha}{\partial x^j}$$

and

$$\tilde{\nabla}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial y^\beta} = \phi^\alpha_i \overset{g'}{\Gamma}{}^\gamma_{\alpha\beta} \frac{\partial}{\partial y^\gamma}$$

where $\overset{g'}{\Gamma}$ are the Christoffel symbols of $\overset{g'}{\nabla}$.

Consider $d\phi = \phi^\alpha_i dx^i \otimes \frac{\partial}{\partial y^\alpha}$ as a smooth section of the vector bundle $T\mathbf{M}^* \otimes \phi^{-1}(T\mathbf{B})$. From Lemma(4.3) the second fundamental form

$$\nabla d\phi(X, Y) = \tilde{\nabla}_X \phi_* Y - \phi_*(\overset{g}{\nabla}_X Y).$$

Also from definition (4.12), ϕ is harmonic map if

$$\tau(\phi) = \text{trace}\nabla d\phi \equiv 0. \tag{4.53}$$

In local coordinates $\tau(\phi)$ can be expressed as

$$\tau(\phi)^\alpha = g^{ij}(\nabla d\phi)^\alpha_{ij}, \tag{4.54}$$

where
$$(\nabla d\phi)^{\alpha}_{ij} = \phi^{\alpha}_{ij} - \phi^{\alpha}_{k}\overset{g}{\Gamma}{}^{k}_{ij} + \overset{g'}{\Gamma}{}^{\alpha}_{\beta\gamma}\phi^{\beta}_{i}\phi^{\gamma}_{j}. \tag{4.55}$$
Here $\phi^{\alpha}_{ij} = \frac{\partial^2\phi^{\alpha}}{\partial x^i \partial x^j}$.

Let $(\mathbf{M}, g)$ and $(\mathbf{B}, g')$ be two Riemannian manifolds of dimensions n and m, respectively. Consider the tangent bundles $T\mathbf{M}$ and $T\mathbf{B}$ endowed with the semi-Riemannian metrics $G = g^c + c^v$ and $H = g'^c + d^v$, respectively, where $c = c_{ij}dx^i dx^j$ and $d = d_{\alpha\beta}dy^{\alpha}dy^{\beta}$ are symmetric tensor fields of type $(0, 2)$ on $\mathbf{M}$ and $\mathbf{B}$, respectively. Let $\overset{G}{\nabla}$, $\overset{H}{\nabla}$ be the Levi-Civita connections of G and H. Denote by $(x^i; u^i)$ for $i = 1 \cdots n$ the local coordinates on $T\mathbf{M}$ induced from the local coordinates (x^i) on $\mathbf{M}$ and by $(y^{\alpha}; v^{\alpha})$ for $\alpha = 1 \cdots m$ the local coordinates on $T\mathbf{B}$ induced from the local coordinates (y^{α}) on $\mathbf{B}$. Let $\phi : \mathbf{M} \longrightarrow \mathbf{B}$ be a smooth map and $\phi_* : T\mathbf{M} \longrightarrow T\mathbf{B}$ be the differential $d\phi$ of ϕ, called the tangent map. The following lemma is on certain properties of the differential of ϕ_*.

Lemma 4.4. *[29] Let $\phi : (\mathbf{M}, g) \longrightarrow (\mathbf{B}, g')$ be a smooth map and $\phi_* : T\mathbf{M} \longrightarrow T\mathbf{B}$ be the induced tangent map. Then, for all vector fields $X \in \mathcal{X}(\mathbf{M})$, we have*
$$\phi_{**}\Big(X^v\Big) = \Big(\phi_*(X)\Big)^v \tag{4.56}$$
$$\phi_{**}\Big(X^h\Big) = \Big(\phi_*(X)\Big)^h + \Big(\nabla d\phi(u, X)\Big), \tag{4.57}$$
where $u = u^i\frac{\partial}{\partial x^i}$ is considered as an element in $T\mathbf{M}$.

Proof. Let (x^i) and (x^i, u^i) be local coordinates on $\mathbf{M}$ and $T\mathbf{M}$, respectively. The local frames of vector fields on $\mathbf{M}$ and $T\mathbf{M}$ are given by $\Big\{\frac{\partial}{\partial x^i} : i = 1, \cdots, n\Big\}$ and $\Big\{\big(\frac{\delta}{\delta x^i}; \frac{\partial}{\partial u^i}\big) : i = 1, \cdots, n\Big\}$, where $\frac{\delta}{\delta x^i} = \big(\frac{\partial}{\partial x^i}\big)^h = \frac{\partial}{\partial x^i} - u^k\overset{g}{\Gamma}{}^{i}_{j,k}\frac{\partial}{\partial u^i}$ and $\frac{\partial}{\partial u^i} = \Big(\frac{\partial}{\partial x^i}\Big)^v$. If (y^{α}) and (y^{α}, v^{α}) are local coordinates on $\mathbf{B}$ and $T\mathbf{B}$ respectively, then the local frames of vector fields on $\mathbf{B}$ and $T\mathbf{B}$ are given by $\Big\{\frac{\partial}{\partial y^{\alpha}} : \alpha = 1, \cdots, m\Big\}$ and $\Big\{\big(\frac{\delta}{\delta y^{\alpha}}; \frac{\partial}{\partial v^{\alpha}}\big) : \alpha = 1, \cdots, m\Big\}$, where $\frac{\delta}{\delta y^{\alpha}} = \big(\frac{\partial}{\partial y^{\alpha}}\big)^h = \frac{\partial}{\partial y^{\alpha}} - v^{\beta}\overset{g'}{\Gamma}{}^{\gamma}_{\alpha,\beta}\frac{\partial}{\partial v_{\gamma}}$ and $\frac{\partial}{\partial v^{\alpha}} = \Big(\frac{\partial}{\partial y^{\alpha}}\Big)^v$. Here $v^{\beta} = u^j\frac{\partial\phi^{\beta}}{\partial x^j} = u^j\phi^{\beta}_{*j}$. Now, we have
$$\phi_{**}\Big(\big(\frac{\partial}{\partial x^i}\big)^h\Big) = \frac{\partial\phi^{\alpha}}{\partial x^i}\frac{\partial}{\partial y^{\alpha}} + u^j\frac{\partial\phi_{*}{}^{\alpha}_{j}}{\partial x^i} - \overset{g}{\Gamma}{}^{k}_{ij}u^j\phi_{*}{}^{\alpha}_{k}\frac{\partial}{\partial v^{\alpha}}. \tag{4.58}$$
$$\phi_{**}\Big(\big(\frac{\partial}{\partial x^i}\big)^v\Big) = \phi_{*}{}^{\alpha}_{i}\frac{\partial}{\partial v^{\alpha}}. \tag{4.59}$$

Since

$$\Big(\phi_*(\frac{\partial}{\partial x^i})\Big)^v = \phi_{*i}^{\ \alpha}(\frac{\partial}{\partial v^\alpha})$$

from (4.59) we get

$$\phi_{**}\Big((\frac{\partial}{\partial x^i})^v\Big) = \Big(\phi_*(\frac{\partial}{\partial x^i})\Big)^v.$$

Hence (4.56) holds. Now consider

$$\frac{\partial \phi^\alpha}{\partial x^i}\frac{\partial}{\partial y^\alpha} = \frac{\partial \phi^\alpha}{\partial x^i}(\frac{\partial}{\partial y^\alpha})^h + \overset{g'}{\Gamma}{}^\gamma_{\alpha\beta} v^\beta \frac{\partial \phi^\alpha}{\partial x^i}\frac{\partial}{\partial v^\gamma}.$$

Substitute this into (4.58) we get

$$\begin{aligned}\phi_{**}\Big((\frac{\partial}{\partial x^i})^h\Big) &= \Big(\phi_*(\frac{\partial}{\partial x^i})\Big)^h + u^j\Big(\frac{\partial \phi_{*j}^{\ \gamma}}{\partial x^i} + \phi_{*j}^{\ \beta}\frac{\partial \phi^\beta}{\partial x^i}\overset{g'}{\Gamma}{}^\gamma_{\alpha\beta} - \overset{g}{\Gamma}{}^k_{ij}\phi_{*k}^{\ \gamma}\Big)(\frac{\partial}{\partial u^\gamma})^v,\\ &= \Big(\phi_*(\frac{\partial}{\partial x^i})\Big)^h + \Big(\nabla d\phi(\frac{\partial}{u, \partial x^i})\Big)^v.\end{aligned}$$

Hence (4.57) holds. □

Remark 4.8. Using the equations (4.56) and (4.57) of above lemma Oproiu [30] proved that $\phi : (\mathbf{M}, g) \longrightarrow (\mathbf{B}, g')$ is an isometric immersion if and only if $\phi_* : (T\mathbf{M}, G) \longrightarrow (T\mathbf{B}, H)$ is an isometric immersion.

Let $\phi_* : (T\mathbf{M}, G) \longrightarrow (T\mathbf{B}, H)$ be the tangent map and $\hat{\nabla}$ denotes the pullback connection of the Levi-Civita connection $\overset{H}{\nabla}$. Then, the second fundamental form of ϕ_* is denoted by $\hat{\nabla} d\phi_*$ and is defined as,

$$\hat{\nabla} d\phi_*(E, F) = \hat{\nabla}_E \phi_* F - \phi_*(\overset{G}{\nabla}_E F),$$

for vector fields $E, F \in \mathcal{X}(T\mathbf{M})$.

Using equations (4.55), (4.56) and (4.57) $\hat{\nabla} d\phi_*$ is represented in local coordinates as

$$\begin{aligned}&(\hat{\nabla} d\phi_*)(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}) = 0,\ \ (\hat{\nabla} d\phi_*)(\frac{\partial}{\partial u^i}, \frac{\delta}{\delta x^j}) = (\hat{\nabla} d\phi_*)(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial u^i}) = (\nabla d\phi)^\alpha_{ij}\frac{\partial}{\partial v^\alpha},\\ &(\hat{\nabla} d\phi_*)(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}) = (\nabla d\phi)^\alpha_{ij}\frac{\delta}{\delta y^\alpha} + \Big[\frac{1}{2}\Psi^\alpha_{ij} + \nabla_0(\nabla d\phi)^\alpha_{ij}\Big]\frac{\partial}{\partial v^\alpha},\end{aligned} \tag{4.60}$$

where

$$\nabla_0(\nabla d\phi)^\alpha_{ij} = u^k\Big[\frac{\partial}{\partial x^k}(\nabla d\phi)^\alpha_{ij} + \phi^\gamma_k \overset{H}{\Gamma}{}^\alpha_{\gamma\beta}(\nabla d\phi)^\beta_{ij} - \overset{G}{\Gamma}{}^h_{ik}(\nabla d\phi)^\alpha_{hj} - \overset{G}{\Gamma}{}^h_{kj}(\nabla d\phi)^\alpha_{ih}\Big]$$

and

$$\Psi^{\alpha}_{ij} = \Big(\overset{g'}{\nabla}_{\gamma} d^{\alpha}_{\beta} + \overset{g'}{\nabla}_{\beta} d^{\alpha}_{\gamma} - \overset{g'}{\nabla}_{\alpha} d^{\gamma}_{\beta}\Big)\phi^{\beta}_{i}\phi^{\gamma}_{j} - \phi^{\alpha}_{k}\Big(\overset{g}{\nabla}_{i} c^{k}_{j} + \overset{g}{\nabla}^{k}_{y} c^{k}_{i} - \overset{g}{\nabla}^{k} c_{ij}\Big).$$

In [30], Oproiu has proved that

Theorem 4.6. *[30] The map $\phi_* : (T\mathbf{M}, G) \longrightarrow (T\mathbf{B}, H)$ is harmonic if and only if $\phi : (\mathbf{M}, g) \longrightarrow (\mathbf{B}, g^{'})$ is harmonic.*

Proof. Consider the matrix of G with respect to the local frame $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})$, That is

$$(G_{ij}) = \begin{bmatrix} c_{ij} & g_{ij} \\ g_{ij} & 0 \end{bmatrix}.$$

Then,

$$(G_{ij})^{-1} = (G^{ij}) = \begin{bmatrix} 0 & g^{ij} \\ g^{ij} & -c_{ij} \end{bmatrix}.$$

Then, by the equation (4.60) the tension field of φ is written as

$$\tau(\phi_*) = 2(\tau(\phi))^{\alpha}\frac{\partial}{\partial v^{\alpha}} = 2(\tau(\phi))^{v}$$

Hence ϕ_* is harmonic if and only if ϕ is harmonic. □

4.3.2 Harmonic Maps between Tangent bundles of Statistical Manifolds

For statistical manifolds we prove that a smooth map $\phi : \mathbf{M} \longrightarrow \mathbf{B}$ is harmonic with respect to ∇ and ∇^* if and only if it is harmonic with respect to the conjugate connections $\overline{\nabla}$ and $\overline{\nabla^*}$. Then, give a necessary condition for the harmonicity of the tangent map with respect to the complete lift structure on the tangent bundle. Also, prove $\pi : (\mathbf{M}, \nabla, g_m) \longrightarrow (\mathbf{B}, \nabla^*, g_b)$ is a statistical submersion if and only if $\pi_* : (T\mathbf{M}, \nabla^c, g^c_m) \longrightarrow (T\mathbf{B}, \nabla^{*c}, g^c_b)$ is a statistical submersion.

Definition 4.15. Let $(\mathbf{M}, \nabla, g_m)$ and $(\mathbf{B}, \nabla^*, g_b)$ be two statistical manifolds. A smooth map $\phi : \mathbf{M} \longrightarrow \mathbf{B}$ is called harmonic with respect to (∇, ∇^*) if the tension field $\tau_{(\nabla,\nabla^*)}(\phi)$ defined with respect to ∇, ∇^* vanishes everywhere.

Now, we have

Theorem 4.7. *Let* $(\mathbf{M}, \nabla, \overline{\nabla}, g_m)$ *and* $(\mathbf{B}, \nabla^*, \overline{\nabla^*}, g_b)$ *be two statistical manifolds and* $\phi : \mathbf{M} \longrightarrow \mathbf{B}$ *be a smooth map. Assume that* ϕ *is harmonic with respect to the Levi-Civita connections* $\overset{g_m}{\nabla}$ *and* $\overset{g_b}{\nabla}$ *of* $\mathbf{M}$ *and* $\mathbf{B}$*, respectively. Then,* ϕ *is harmonic with respect to* ∇, ∇^* *if and only if it is harmonic with respect to* $\overline{\nabla}, \overline{\nabla^*}$*.*

Proof. Since,

$$\frac{\nabla + \overline{\nabla}}{2} = \overset{g_m}{\nabla} \quad \text{and} \quad \frac{\nabla^* + \overline{\nabla^*}}{2} = \overset{g_b}{\nabla} \tag{4.61}$$

we have

$$\overset{g_b}{\nabla}_{\phi_* X}\phi_* Y - \phi_*(\overset{g_m}{\nabla}_X Y) = \frac{1}{2}\Big[\nabla^*_{\phi_* X}\phi_* Y - \phi_*(\nabla_X Y)\Big] + \frac{1}{2}\Big[\overline{\nabla^*}_{\phi_* X}\phi_* Y - \phi_*(\overline{\nabla}_X Y)\Big]. \tag{4.62}$$

Then,

$$2\tau_{\left(\overset{g_m}{\nabla},\overset{g_b}{\nabla}\right)}(\phi) = \tau_{(\nabla,\nabla^*)}(\phi) + \tau_{(\overline{\nabla},\overline{\nabla^*})}(\phi). \tag{4.63}$$

Since ϕ is harmonic with respect to $\overset{g_m}{\nabla}$ and $\overset{g_b}{\nabla}$ we have

$$\tau_{(\nabla,\nabla^*)}(\phi) = -\tau_{(\overline{\nabla},\overline{\nabla^*})}(\phi). \tag{4.64}$$

Hence, ϕ is harmonic with respect to ∇, ∇^* if and only if it is harmonic with respect to $\overline{\nabla}, \overline{\nabla^*}$ □

Consider a statistical manifold $(\mathbf{M}, \nabla, g_m)$, we have $(T\mathbf{M}, \nabla^c, g_m^c)$ is also a statistical manifold (cf. Proposition (4.2)), where ∇^c and g_m^c are the complete lift of ∇ and g_m, respectively. In the following theorem we prove a necessary condition for the harmonicity of the tangent map with respect to the complete lift structure on the tangent bundles.

Theorem 4.8. *Let* $(\mathbf{M}, \nabla, g_m)$ *and* $(\mathbf{B}, \nabla^*, g_b)$ *be two statistical manifolds and* $\phi : \mathbf{M} \longrightarrow \mathbf{B}$ *be a harmonic map. Then, the tangent map* $\phi_* : (T\mathbf{M}, \nabla^c, g_m^c) \longrightarrow (T\mathbf{B}, \nabla^{*c}, g_b^c)$ *is harmonic with respect to* ∇^c, ∇^{*c} *if* $\phi_{**}(X^c) = (\phi_*(X))^c$ *for* $X \in \mathcal{X}(\mathbf{M})$*.*

Proof. By the definition of complete lift

$$\nabla^{*c}_{\phi_{**}X^c}\phi_{**}Y^c - \phi_{**}(\nabla^c_{X^c}Y^c) = \nabla^{*c}_{\phi_{**}X^c}\phi_{**}Y^c - \phi_{**}((\nabla_X Y)^c). \tag{4.65}$$

Since $\phi_{**}(X^c) = (\phi_*(X))^c$, we get

$$\begin{aligned}
\nabla^{*c}_{\phi_{**}X^c}\phi_{**}Y^c - \phi_{**}(\nabla^c_{X^c}Y^c) &= \nabla^{*c}_{(\phi_*X)^C}(\phi_*Y)^C - (\phi_*(\nabla_X Y))^c \qquad (4.66)\\
&= \Big(\nabla^*_{\phi_*X}\phi_*Y - \phi_*(\nabla_X Y)\Big)^c. \qquad (4.67)
\end{aligned}$$

Now, since ϕ is harmonic we get $\tau_{(\nabla^c,\nabla^{*c})}(\phi_*) = 0$. That is, ϕ_* is harmonic with respect to ∇^c, ∇^{*c}. $\square$

Recall the definition of the statistical submersion.

Definition 4.16. Let $(\mathbf{M}, \nabla, g_m)$ and $(\mathbf{B}, \nabla^*, g_b)$ be two statistical manifolds. Then, a semi-Riemannian submersion $\pi : \mathbf{M} \to \mathbf{B}$ is said to be a statistical submersion if

$$\pi_*(\nabla_X Y)_p = (\nabla^*_{X'} Y')_{\pi(p)},$$

for basic vector fields X, Y on $\mathbf{M}$ which are π-related to X' and Y' on $\mathbf{B}$ and $p \in \mathbf{M}$.

Remark 4.9. Note that every statistical submersion is a harmonic map.

Now, we have

Proposition 4.8. *Let $(\mathbf{M}, \nabla, g_m)$ and $(\mathbf{B}, \nabla^*, g_b)$ be two statistical manifolds and $\pi : \mathbf{M} \to \mathbf{B}$ be a smooth such that $\pi_{**}(X^c) = (\pi_*(X))^c$. Then π is a statistical submersion if and only if $\pi_* : (T\mathbf{M}, \nabla^c, g^c_m) \longrightarrow (T\mathbf{B}, \nabla^{*c}, g^c_b)$ is a statistical submersion.*

Proof. Assume that π is a statistical submersion. Now, by the definition of the complete lift

$$\pi_{**}(\nabla^c_{X^c}Y^c) = \pi_{**}((\nabla_X Y)^c).$$

Since, $\pi_{**}(X^c) = (\pi_*(X))^c$ we get

$$\begin{aligned}
\pi_{**}(\nabla^c_{X^c}Y^c) &= \Big[\pi_*(\nabla_X Y)\Big]^c\\
&= \Big[\nabla^*_{\pi_*X}\pi_*Y\Big]^c \quad \text{(since, } \pi \text{ is a statistical submersion)}\\
&= \nabla^{*c}_{\pi_{**}X^c}\pi_{**}Y^c
\end{aligned}$$

Hence, π_* is a statistical submersion.

Conversely, assume that π_* is a statistical submersion. Then

$$\pi_{**}(\nabla^c_{X^c} Y^c) \quad = \quad \nabla^{*c}_{\pi_{**} X^c} \pi_{**} Y^c. \tag{4.68}$$

Since, $\pi_{**}(X^c) = (\pi_*(X))^c$ from (4.68) we get

$$\pi_{**}((\nabla_X Y)^c) \quad = \quad (\nabla^*_{\pi_* X} \pi_* Y)^c.$$

That is,

$$(\pi_*(\nabla_X Y))^c \quad = \quad (\nabla^*_{\pi_* X} \pi_* Y)^c.$$

Then $\pi_*(\nabla_X Y) = \nabla^*_{\pi_* X} \pi_* Y$, hence π is a statistical submersion. $\square$

Chapter 5

Geometry of Estimation in Statistical Manifolds

In [31], Amari discussed the statistical properties of an estimator in a statistical manifold and the geometry of exponential and curved exponential family. Amari [31] proved a necessary and sufficient condition for a submanifold of an exponential family to be exponential. Amari has obtained geometric conditions for the consistency and efficiency of an estimator in a curved exponential family using ancilliary manifolds [33], [31]. Cheng et al. [34] obtained an MLE algorithm for estimating parameters in the curved exponential family. The estimator belongs to the gradient-based methods that operate on statistical manifolds.

A statistical model is a family M of probability measures on a measurable space (sample space) Ω. In the case of a finite dimensional parametrized families of measures, the theory of statistical manifold structure with the dual connections is well studied. Also a statistical manifold - a Riemannian manifold with each of whose points is a probability distribution - can be embedded into the space of probability measures on a finite set. Infinite dimensional families of probability measures were first considered by Pistone and Sempi [35]. To deal with the infinite dimensional spaces of probability measures Ay et al.[36] developed a functional analytic framework. They introduced the notion of parametrized measure models and obtained the analogue of the structures considered in the finite dimensional information geometry.

In this chapter, estimation of parameters in statistical manifolds, submanifolds of exponential family, estimation of parameters in the curved exponential family and Fisher-Neyman sufficient statistic for parametrized model are discussed. In section 5.1, short account of the statistical properties of an estimator is given. In section 5.2, we show that if all ∇^1- autoparallel proper submanifolds of a ± 1- flat statistical manifold M are exponential then M is an exponential family. Also, we show that if a submanifold of a statistical model is an exponential family, then it is a ∇^1-autoparallel submanifold [32]. We prove that the Fisher-Neyman sufficient statistic is invariant under the isostatistical immersions of statistical manifolds in section 5.3.

5.1 Estimation of Parameters in Statistical Manifolds

In this section, a brief description of the statistical properties of an estimator in a statistical manifold is given [1].

Let $\mathbf{M} = \{p(x; \theta) : \theta \in \Theta \subset \mathbb{R}^n\}$ be an n-dimensional statistical model parametrized by $\theta = [\theta^i]$, for $i = 1, 2, ...n$. This may be viewed as an n-dimensional statistical manifold under appropriate regularity conditions.

Definition 5.1. Let $\mathbf{M} = \{p(x, \theta) : \theta \in \Theta \subseteq \mathbb{R}^n\}$ be an n-dimensional statistical manifold and $g = \langle . \rangle$ be the Fisher information metric. Let $\alpha \in \mathbb{R}$.
Define n^3 functions

$$\Gamma^{\alpha}_{ij,k} = E_\theta[(\partial_i \partial_j \ell_\theta + \frac{1-\alpha}{2} \partial_i \ell_\theta \partial_j \ell_\theta) \partial_k \ell_\theta]$$

Then an affine connection ∇^α on $\mathbf{M}$ is defined by $\langle \nabla^\alpha_{\partial_i} \partial_j, \partial_k \rangle = \Gamma^\alpha_{ij,k}$, known as Amari's α- connection.

Note. If $\alpha = 1$, then ∇^1 is called an exponential connection or e-connection. Also, note that ∇^α is flat if and only if $\nabla^{-\alpha}$ is flat.

Let $x_N = (x^1, \cdots, x^N)$ be N independent and identically distributed random variables drawn from $p(x; \theta)$. Then, the joint density of x_N is

$$P_N(x_N; \theta) \quad = \quad \prod_{t=1}^{N} p(x^t; \theta).$$

Also, the log-likelihood of the density is

$$\log P_N(x_N; \theta) \quad = \quad \sum_{t=1}^{n} \log p(x^t; \theta).$$

By viewing x_N as a random variable, consider $\mathbf{M}_N = \{P_N(x_N; \theta) : \theta \in \Theta \subset \mathbb{R}^n\}$ as an n-dimensional statistical manifold with θ as the coordinate system. Then, the Fisher information and Amari's α- connection of $\mathbf{M}_N$ in local coordinates are

$$\begin{aligned} g^N_{ij}(\theta) &= N g_{ij}(\theta), \\ \Gamma^{(\alpha)N}_{ij,k} &= N \Gamma^{(\alpha)}_{ij,k}, \end{aligned}$$

where $g_{ij}(\theta)$ is the component of Fisher information metric on $\mathbf{M}$ and $\Gamma^{(\alpha)}_{ij,k}$ is the component of α-connection. Note that the geometry of $\mathbf{M}_N$ is the same as that of $\mathbf{M}$ scaled by a factor N. So the natural basis for the tangent vectors ∂^N_i of $\mathbf{M}_N$ are given by $\partial^N_i = \sqrt{N}\partial_i$, where $\partial_i = \frac{\partial}{\partial \theta^i}$.

One of the main objective of the estimation problem is to select an appropriate estimator such that its sampling distribution is concentrated around the actual value of the unknown parameter θ. An estimator $\hat{\theta}_N$ is defined as a function of the data point x_N given by

$$\hat{\theta}_N = \hat{\theta}_N(x_N) = \hat{\theta}_N(x^1, x^2, ...x^N).$$

Note that the estimator $\hat{\theta}_N$ depends on N, but for the notational convenience we denote $\hat{\theta}_N$ by $\hat{\theta}$.

Estimation error is given by $e = \hat{\theta} - \theta$ and bias of the estimator is given by $b(\theta) = E_\theta[\hat{\theta}] - \theta$, where E_θ is the expectation with respect to the distribution $P_N(x_N;\theta)$. An estimator is said to be **unbiased** if $b(\theta) = 0$. The **mean square error** of an estimator is expressed as the matrix

$$MSE(\hat{\theta}) = \left[E_\theta \left[(\hat{\theta}^i - \theta^i)(\hat{\theta}^j - \theta^j) \right] \right].$$

The accuracy of an estimator is measured by the **variance-covariance matrix** $V_\theta(\hat{\theta}) = [v^{ij}_\theta(\hat{\theta})]$, where

$$v^{ij}_\theta(\hat{\theta}) \;\; = \;\; E_\theta[(\hat{\theta}_i - E[\hat{\theta}_i])(\hat{\theta}_j - E[\hat{\theta}_j])].$$

Note that, if the estimator $\hat{\theta}$ is unbiased then the mean square error is the variance-covariance matrix of the estimator. That is, $MSE(\hat{\theta}) = V_\theta(\hat{\theta})$.

The **Cramer- Rao inequality** $[v^{ij}_\theta(\hat{\theta})] \geq \frac{1}{N}[g^{ij}]$, where g^{ij} is the inverse of Fisher information metric and $\hat{\theta}$ is an unbiased estimator, gives a bound of accuracy. An unbiased estimator $\hat{\theta}$ which achieves Cramer-Rao equality $[v^{ij}_\theta(\hat{\theta})] = \frac{1}{N}[g^{ij}]$ is called the **finite sample efficient estimator**.

So far, we have described the properties of an estimator obtained by a sample of size

N. The asymptotic theory studies the behaviour of an estimator when N is large. For describing the finite sample theory the estimator $\hat{\theta}_N$ is denoted by $\hat{\theta}$. Let $\{\hat{\theta}_N, N = 1, 2, ...\}$ denotes the estimator for asymptotic analysis.

An estimator $\{\hat{\theta}_N, N = 1, 2, ...\}$ is said to be **consistent** if for all θ the estimator $\hat{\theta}_N(x_N)$ converges in probability to θ (denoted by $\hat{\theta}_N(x_N) \xrightarrow{p} \theta$) as $N \longrightarrow \infty$. That is, for all θ and for every $\epsilon > 0$,

$$\lim_{N \longrightarrow \infty} Pr_\theta\{| \hat{\theta}_N - \theta |> \epsilon\} = 0.$$

The concept of mean consistency is a far more powerful condition than the usual notion of consistency. An estimator $\hat{\theta}_N$ is said to be **mean consistent** if

$$\lim_{N \longrightarrow \infty} E_\theta[\hat{\theta}_N] = \theta, \quad \lim_{N \longrightarrow \infty} \partial_j E_\theta[\hat{\theta}^i_N] = \partial_j \theta^i = \delta_{ij}.$$

This type of estimator is often known as an **asymptotically unbiased estimator**.

Remark 5.1. In [33], Amari proved that the coordinate system (θ_i) of the statistical model $\mathbf{M} = \{p(x, \theta)\}$ has an efficient estimator if and only if $\mathbf{M}$ is an exponential family and θ is m-affine. For more details about asymptotic analysis on statistical manifolds refer, [33], [1].

5.2 Exponential Family and Estimation Theory

In this section, we first discuss about the geometry of exponential family. In [1], Amari and Nagaoka obtained a necessary and sufficient condition for a submanifold of an exponential family to be exponential. We show that if all ∇^1-autoparallel proper submanifolds of a ± 1-flat statistical manifold $\mathbf{M}$ are exponential then $\mathbf{M}$ is an exponential family. Also, we show that if submanifold of a statistical model is an exponential family, then it is a ∇^1-autoparallel submanifold [32]. Then a brief account of Amari's geometric conditions for the consistency and efficiency of an estimator in a curved exponential family using ancilliary manifold is given [33], [31]. Also, discussed the work of Cheng et al. [34] on an iterative maximum likelihood estimator in a curved exponential family.

Let $(\Omega, \mathcal{B}, \mu)$ be a measure space. A parametrized model $\mathbf{M} = \{p(x; \theta) : \theta \in \Theta \subset \mathbb{R}^n\}$ is called an n-dimensional exponential family, if

$$p(x, \theta) = \exp(\sum_{i=1}^{n} \theta^i F_i(x) + C(x) - \psi(\theta)), \tag{5.1}$$

where $\theta = \{\theta^1, \cdots, \theta^2\}$ is an n-dimensional vector parameter called natural parameter, $\{C, F_i\}$ are functions on Ω. Here ψ corresponds to the normalization factor. The expectation parameter η_i is defined as

$$\eta_i = E_\theta[F_i] = \int_\Omega F_i(x)p(x;\theta)dx. \tag{5.2}$$

This is also called the dual parameter of θ. In [31], Amari proved that $\mathbf{M}$ is a dually flat space with dual coordinate systems θ and η. Note that the n functions $F_1(x) \cdots F_n(x)$ are random variables, rename the n random variables as follows

$$x_i = F_i(x) \quad (i = 1, \cdots, n).$$

Now, define the probability density function on the random variable $x = \{x_1, x_2, ...x_n\}$ with respect to the dominating measure

$$d\mu(x) = \text{exp}\{C(x)\}dx.$$

Then, the equation (5.1) can be written as

$$p(x, \theta) = \exp(\sum_{i=1}^n \theta^i x_i - \psi(\theta)).$$

Collection of these distributions is referred as the standard exponential family. Note that the exponential family is an n-dimensional statistical manifold.

Note. In [31], Amari proved that the exponential family is ± 1-flat, (that is, flat with respect to $\nabla^{\pm 1}$ connection). Also, note that the parametrized model which is flat with respect to ± 1 connection need not be an exponential family.

Example 5.1. Let q be a smooth probability density function on $\mathbf{R}$ and q^k be the k^{th} independent and identically distributed extension. Then, for

$$y = (y^1, y^2, y^3, ...y^k)^t \tag{5.3}$$

we have

$$q^k(Y) = q(y^1)q(y^2)q(y^3), ...q(y^k). \tag{5.4}$$

For a regular matrix $A \in \mathbb{R}^{k \times k}$ and a vector $v \in \mathbb{R}^k$, define the probability density function on $\mathbb{R}^k$ by

$$p(A, v, x) = \frac{q^k(A^{-1}(x - v))}{\mid det(A) \mid}, \tag{5.5}$$

which gives the probability distribution for the random variable $AY + v$ when Y is distributed according to $q^k(y)$. Now, define a statistical model

$$S = \{p(A, v, x) : v \in \mathbb{R}^k\}. \tag{5.6}$$

Then we can easily see that $\frac{\partial log(p(A,v,x))}{\partial v_i}$ is constant. So from the definition of Amari's α-connection, $\Gamma^{\alpha}_{ij,k}$ = 0 and it implies that S is α-flat for all α. But in general S is not an exponential family.

Definition 5.2. Let $(\mathbf{M}, \nabla, g_m)$ be an n-dimensional statistical manifold. A submanifold $\mathbf{S}$ of $\mathbf{M}$ is said to be ∇-autoparallel if $\nabla_X Y \in \mathcal{X}(\mathbf{S})$ for $X, Y \in \mathcal{X}(\mathbf{S})$.
1-dimensional autoparallel submanifolds are called geodesics.

Remark 5.2. A necessary and sufficient condition for $\mathbf{S}$ to be autoparallel is that $\nabla_{\partial a}\partial b \in \mathcal{X}(\mathbf{S})$ holds for $a, b = 1, \cdots, m$, where m is the dimension of $\mathbf{S}$.

Let $(\mathbf{M}, \nabla, g_m)$ be an n-dimensional statistical manifold and $\mathbf{S}$ be an m-dimensional submanifold of $\mathbf{M}$ then $\mathrm{T}_p\mathbf{S} \subset \mathrm{T}_p\mathbf{M}$ for $p \in \mathbf{S}$. Now, consider the orthogonal projection $\pi_p : \mathrm{T}_p\mathbf{S} \longrightarrow \mathrm{T}_p\mathbf{M}$ and $\pi_p(D)$ = D, for all $D \in \mathrm{T}_p\mathbf{S}$. Define a connection ∇^π on $\mathbf{S}$ as

$$(\nabla^\pi_X Y)_p = \pi_p(\nabla_X Y)_p,$$

for all $p \in \mathbf{S}$. Also, define

$$H(X, Y) = \nabla_X Y - \nabla^\pi_X Y.$$

$H(X, Y)$ is called the second fundamental form or the embedding curvature. Now for each $p \in \mathbf{M}$, let $\{(\partial_a)_p; 1 \leq a \leq m\}$ be a basis for $\mathrm{T}_p\mathbf{S}$ and let $\{(\partial_k)_p; m + 1 \leq k \leq n\}$ be a basis for $\mathrm{T}_p\mathbf{S}^\perp$. Then, define $m^2(n - m)$ functions $\{H_{abk}\}$ in the following way

$$H_{abk} = g_m\bigg(H(\partial_a, \partial_b), \partial_k\bigg) = g_m\bigg(\nabla_{\partial_a}\partial_b, \partial_k\bigg).$$

It follows that $H = 0$ if and only if $H_{abk} = 0$ for all a, b, k. Also, note that $H(X, Y)$ = 0 if and only if $\mathbf{S}$ is ∇-autoparallel submanifold of $\mathbf{M}$.

Remark 5.3. In [1], Amari and Nagaoka proved that a submanifold $\mathbf{S}$ is autoparllel in a flat manifold $\mathbf{M}$ if and only if $\mathbf{S}$ can be expressed as an affine subspace of $\mathbf{M}$ with respect to an affine coordinate system. Also proved that if $\mathbf{S}$ is autoparallel, then it is also flat.

Amari and Nagaoka has proved a necessary and sufficient condition for a submanifold of an exponential family to be exponential.

Theorem 5.1. *[1] Let* $\mathbf{M}$ *be an exponential family and* $\mathbf{S}$ *be a submanifold of* $\mathbf{M}$*. Then,* $\mathbf{S}$ *is an exponential family if and only if* $\mathbf{S}$ *is* ∇^1*-autoparallel in* $\mathbf{M}$*.*

Proof. Assume that $\mathbf{S}$ is autoparallel with respect to ∇^1 in $\mathbf{M}$. Then by remark (5.3), $\mathbf{S}$ is an exponential family. Conversely, assume that $\mathbf{S}$ is an exponential family. Let $\mathbf{M} = \{p(x,\theta) : \theta \in \Theta \subset \mathbf{R}^n\}$ and $\mathbf{S} = \{q(x,u) : u \in U \subset \mathbf{R}^m\}$. By definition of the exponential family,

$$p(x,\theta) = \exp(\sum_{i=1}^{n} \theta^i F_i(x) + C(x) - \psi(\theta))$$

and

$$q(x,u) = p(x,\theta(u)) = \exp\{\sum_{a=1}^{m} u^a G_a(x) + D(x) - \phi(u)\}.$$

Then

$$\begin{aligned} G_a(x) - \partial_a \phi(u) &= \partial_a \log(q(x,u)) \\ &= (\partial_a \theta^i)_u \partial_i \log(p(x,\theta(u))) \\ &= (\partial_a \theta^i)_u \{F_i(x) - \partial_i \psi(\theta(u))\}, \end{aligned}$$

and hence

$$(\partial_a \theta^i)_u F_i(x) + \lambda_a(u) = G_a(x),$$

where $\lambda_a(u)$ is constant with respect to x. Since $G_a(x)$ does not depend on u and since $\{F_1, ..., F_n, 1\}$ are assumed to be linearly independent, $(\partial_a \theta^i)_u$ is constant with respect to u for all i and a. Then, from remark (5.3) we get $\mathbf{S}$ is autoparallel with respect to ∇^1 in $\mathbf{M}$. □

Now, we show that if all ∇^1-autoparallel proper submanifolds of a ± 1-flat statistical manifold $\mathbf{M}$ are exponential, then $\mathbf{M}$ is an exponential family.

Theorem 5.2. *Let* $\mathbf{M} = \{p(x,\theta) : \theta \in \Theta\}$ *be a parametrized family which is flat with respect to* ∇^1 *and* ∇^{-1}*. If all* ∇^1*-autoparallel submanifolds of* $\mathbf{M}$ *are exponential, then* $\mathbf{M}$ *is an exponential family.*

Proof. Let $\mathbf{M} = \{p(x,\theta) : \theta \in \Theta\}$, be an n-dimensional statistical manifold with dually flat structure $(g, \nabla^1, \nabla^{-1})$, where g is the fisher information metric. Let $\theta = [\theta^i]$ and $\eta = [\eta_j]$

be the coordinate systems of $\mathbf{M}$ with respect to ∇^1 and ∇^{-1} respectively. Now, subdivide the range of index $i = 1, 2, ...n$ into two indexing sets $I = \{i = 1, 2, ...k\}$ and $II = \{i = k+1, k+2, ...n\}$. Let $M(C_{II})$ be the set of points whose coordinates $[\theta^i]$ in II are fixed to the constants $C_{II} = (C^i_{II})$ for $i = k+1, k+2, ..n$. That is,

$$M(C_{II}) = \{p \in \mathbf{M} : \theta^{k+1} = C^{k+1}_{II}, \theta^{k+2} = C^{k+2}_{II}, ...\theta^n = C^n_{II}\}, \tag{5.7}$$

where $C_{II} \in \mathbb{R}^{n-k}$. Then, this is an affine space with respect to θ-coordinate system, which implies $M(C_{II})$ is a ∇^1-autoparallel submanifold of $\mathbf{M}$. Also, if $C_{II} \neq C'_{II}$, then $M(C_{II}) \cap M(C'_{II}) = \phi$ and $\bigcup_{C_{II}} M(C_{II}) = M$. Now, by our assumption $M(C_{II})$ is an exponential family for all C_{II}. If $p(x,\theta) \in \mathbf{M}$, then $p(x,\theta) \in M(C_{II})$ for some constant C_{II}, this implies that

$$p(x,\theta) = exp(\sum_{i=1}^{k} \theta^i x_i - \psi^\beta(\theta)), \tag{5.8}$$

where $\psi^\beta(\theta)$ defined on $\Theta^\beta = \{\theta \in \Theta \mid \theta^{k+1} = C^{k+1}_{II}, \theta^{k+2} = C^{k+2}_{II}, ...\theta^n = C^n_{II}\}$. Now define $\phi(\theta) = \psi^\beta(\theta)$ if $\theta \in \Theta^\beta$. Then, we have

$$\begin{aligned} p(x,\theta) &= exp(\sum_{i=1}^{k} \theta^i x_i - \phi(\theta)) && (5.9)\\ &= exp(\sum_{i=1}^{k} \theta^i x_i + \sum_{i=k+1}^{n} C^i_{II} x_i - \sum_{i=1}^{k} C^i_{II} x_i - \phi(\theta)) && (5.10)\\ &= exp(\sum_{i=1}^{n} \theta^i x_i + F(x) - \phi(\theta)), && (5.11) \end{aligned}$$

where $F(x) = -\sum_{i=1}^{k} C^i_{II} x_i$ for $p(x,\theta) \in M(C_{II})$, then $\mathbf{M}$ is an exponential family. □

Now, we show that if submanifold of a statistical model is an exponential family, then it is a ∇^1-autoparallel submanifold.

Theorem 5.3. *Let* $\mathbf{M} = \{p(x,\theta) : \theta \in \Theta\}$ *be a statistical manifold with* ∇^1 *connection and* $\mathbf{S}$ *be a submanifold of* $\mathbf{M}$. *If* $\mathbf{S}$ *is an exponential family, then* $\mathbf{S}$ *is* ∇^1*-autoparallel submanifold of* $\mathbf{M}$.

Proof. Let $\mathbf{M} = \{p(x,\theta) \mid \theta \in \Theta\}$ and $\mathbf{S} = \{q(x,u)\}$ be a submanifold of $\mathbf{M}$. Let $[\theta^i]$ and

$[u_a]$ be coordinates of $\mathbf{M}$ and $\mathbf{S}$, respectively. Suppose $\mathbf{S}$ is an exponential family, then

$$q(x,u) = p(x,\theta(u)) = \exp\{\sum_{a=1}^{n} u_a G^a(x) + D(x) - \phi(u)\}. \tag{5.12}$$

We have,

$$\Gamma^1_{ab,k} = E_\xi[(\partial_a \partial_a \ell_\theta)\partial_k \ell_\theta],$$

where ℓ_θ =$log(p(x,\theta))$. Then

$$\partial_a \partial_a \ell_\theta = -\frac{\partial^2 \phi}{\partial u_a \partial u_b}.$$

Therefore, $\Gamma^1_{ab,k} = 0$ which implies $\langle \nabla^1_{\partial_a} \partial_b, \partial_k \rangle = 0$, for all k. Hence, $H_{abk} = 0$, which implies that $\mathbf{S}$ is a ∇^1-autoparallel submanifold of $\mathbf{M}$. □

Note that in the above theorem we are not assuming that the ambient manifold $\mathbf{M}$ is an exponential family. Next, we discuss the estimation theory in exponential family.

Let $x_N = \{x^1, x^2, ...x^N\}$ be independent and identically distributed random variables drawn from $p(x,\theta) \in \mathbf{M}$. Then

$$\begin{aligned} P(x_N;\theta) &= \prod_{t=1}^{N} p(x^t;\theta) \\ &= \prod_{t=1}^{N} \exp(\sum_{i=1}^{n} \theta^i x_i^t - \psi(\theta)) \\ &= \exp(N\{\theta \cdot \overline{x} - \psi(\theta)\}), \end{aligned}$$

where $\overline{x} = (\overline{x}_1, \cdots, \overline{x}_n)$ is the arithmetic mean given by

$$\overline{x}_i = \frac{x_i^1 + \cdots + x_i^N}{N}, \qquad i = 1, 2, \cdots, n.$$

This shows that $\{p_N(x_N;\theta)\}$ is also an exponential family with the natural parameters (θ^i). Also, note that the joint probability density $p_N(x_N;\theta)$ depends on the N observations $\{x^1, \cdots, x^N\}$ through $\overline{x}$. Thus the statistic $\overline{x}$ is a sufficient statistic for θ and is called the observed point.

Now take $\hat{\eta}_N = \overline{x}$ as an estimator for η. Then,

$$\begin{aligned} E_\theta[\overline{x}] &= \eta. \\ E_\theta[(\overline{x}_i - \eta_i)(\overline{x}_j - \eta_j)] &= \frac{1}{N} g_{ij}(\theta). \end{aligned}$$

Therefore, $\hat{\eta}_N = \overline{x}$ is an unbiased and efficient estimator for η. As a result, a finite dimensional standard exponential family has a sufficient statistic and an efficient estimate [1], [38].

Definition 5.3. Let $\mathbf{M} = \{p(x;\theta) : \theta \in \Theta \subset \mathbb{R}^n\}$ be an n-dimensional exponential family. A subfamily $\mathbf{S} = \{p(x;\theta(u)) : u \in U \subset \mathbb{R}^m\}$ of $\mathbf{M}$ is called a curved exponential family with parameter u if

$$p(x;\theta(u)) = \exp\left(\sum_{i=1}^{m} \theta^i(u) x_i - \psi(\theta(u))\right).$$

Let $x_N = (x^1, x^2, ...x^N)$ be N independent and identically distributed observations drawn from $p(x;\theta(u)) \in \mathbf{S}$. This gives an observed point $\overline{x} = (\overline{x}_1, \cdots, \overline{x}_n)$ in the exponential family $\mathbf{M}$ and defines a distribution in $\mathbf{M}$ whose η coordinate is given by $\hat{\eta}_N = \overline{x}$. But this point need not be in $\mathbf{S}$. Now the estimate of u is obtained by mapping $\hat{\eta}_N$ to $\mathbf{S}$. That is, $f_N(\hat{\eta}_N) = \hat{u}_N$ is the estimator for u, where $f_N : \mathbf{M} \longrightarrow \mathbf{S}$. This map f_N is known as the estimator. Note that this estimator is depending upon the sample size N.

Definition 5.4. Let $f_N : \mathrm{M} \longrightarrow \mathbf{S}$ be an estimator. Define

$$\mathcal{A}_N(u) \;\; = \;\; \{\eta \in M : f_N(\eta) = u\},$$

which is an $(n-m)$-dimensional submanifold of $\mathbf{M}$. It is called an estimating submanifold or an ancillary manifold.

Let

$$A(u) = \lim_{N \longrightarrow \infty} A_N(u).$$

Assume that each f_N is a continuous function from $\mathbf{M}$ to $\mathbf{S}$ and f be the limiting estimator function which determines the limiting estimating submanifold $A(u)$.

Remark 5.4. Let $\mathbf{S} = \{p(x;\theta(u)) : u \in U \subset \mathbb{R}^m\}$ be an m-dimensional curved subfamily of $\mathbf{M}$. In [60], Amari's geometric conditions on consistency and first order efficiency of an estimator in a curved exponential family are given as

- An estimator $\{\hat{u}_N, N = 1, 2, ...\}$ for $u \in \mathbf{S}$ is consistent if and only if $\eta(u) \in \mathbf{S}$ is in the estimating submanifold $A(u)$.

- A consistent estimator $\{\hat{u}_N, N = 1, 2, ...\}$ for $u \in \mathbf{S}$ is first order efficient if and only if $A(u)$ is orthogonal to $\mathbf{M}$ at the intersecting point $\eta(u)$ in $\mathbf{S}$.

Let $\mathbf{M} = \{p(x;\theta) : \theta \in \Theta \subset \mathbb{R}^n\}$ be an n-dimensional exponential family and $\mathbf{S} = \{p(x;\theta(u)) : u \in U \subset \mathbb{R}^m\}$ be an m-dimensional curved exponential family of $\mathbf{M}$. The Kullback-Leibler divergence, $D(. \parallel .)$, on $\mathbf{M}$ is defined as

$$D(p(x;\theta) \parallel q(x;\theta)) = \int p(x;\theta) \log \left(\frac{p(x;\theta)}{q(x;\theta)}\right) dx.$$

Let $\hat{\eta}_N$ be the observed point with respect to the independent and identically distributed sample $x_N = (x^1, \cdots, x^N)$ drawn from $p(x;\theta(u)) \in \mathbf{S}$. Then the Kullback-Leibler divergence from the observed point $\hat{\eta}$ to a point $\eta(u)$ in $\mathbf{S}$ is,

$$\begin{aligned} D(\hat{\eta}_N \parallel \eta(u)) &= \psi(\theta(u)) + \varphi(\hat{\eta}) - \theta^i(u)\hat{\eta}_{Ni} \\ &= \varphi(\hat{\eta}_N) - \frac{1}{N} \log P_N(x_N; u), \end{aligned}$$

where ψ and φ are potential functions of the exponential family with $\frac{\partial \psi}{\partial \theta^i} = \eta_i$ and $\frac{\partial \varphi}{\partial \eta_i} = \theta^i$, respectively.

Remark 5.5. In [60], Amari proved that the point which minimizes the divergence was the orthogonal projection of the point $\hat{\eta}$ onto S along a ∇^{-1}-geodesic.

Definition 5.5. Let $\mathrm{M} = \{p(x;\theta) : \theta \in \Theta \subset \mathbb{R}^n\}$ be an n-dimensional statistical manifold. Consider N independent and identically distributed random variables $x_N = \{x^1, ...x^N\}$ drawn from $p(x;\theta)$. Then, the **likelihood function** $L^N(\theta)$ is given by

$$L^N(\theta) = P_N(x_N;\theta) = \prod_{t=1}^{N} p(x^t;\theta). \tag{5.13}$$

Since $\log$ function is a strictly increasing function, maximizing the likelihood function $L^N(\theta)$ is equivalent to maximizing $\log(L^N(\theta))$. The estimator $\hat{\theta}$ is said to be the **Maximum Likelihood Estimator (MLE)** denoted by $\hat{\theta}_{ML}$ if

$$\hat{\theta} = arg \max_{\theta \in \Theta} L^N(\theta) = arg \max_{\theta \in \Theta} \sum_{t=1}^{N} \log(p(x^t;\theta))$$

Proposition 5.1. *[1] The maximum likelihood estimator $\hat{u}_{ML}$ of the parameter u of the curved exponential family $\mathbf{S} = \{p(x; \theta(u)) : u \in U \subset \mathbb{R}^m\}$ is an asymptotically efficient estimator.*

Proof. Since

$$\begin{aligned} D(\hat{\eta}_N \parallel \eta(u)) &= \psi(\theta(u)) + \varphi(\hat{\eta}) - \theta^i(u)\hat{\eta}_{Ni} \\ &= \varphi(\hat{\eta}_N) - \frac{1}{N}\log P_N(x_N; u). \end{aligned}$$

The point which minimizes the divergence with point $\hat{\eta}_N$ is the point which maximizes the likelihood $P_N(x_N; u)$, and this is the maximum likelihood estimator $\hat{u}_{ML}$. Hence from remark (5.5), the estimating submanifold $A(u)$ of $\hat{u}_{ML}$ is autoparallel with respect to ∇^{-1} and orthogonal to $\mathbf{S}$. Hence from remark (5.4) $\hat{u}_{ML}$ is asymptotically efficient. $\square$

In [34], Cheng et al. developed an algorithm to obtain the maximum likelihood estimator for the parameter u of the curved exponential family $\mathbf{S}$ using natural gradient descent on statistical manifolds. Now, we discuss the geometry of this algorithm in detail.

Let $\mathbf{S} = \{p(x; \theta(u)) : u \in U \subset \mathbb{R}^m\}$ be a curved exponential family. That is,

$$p(x; u) = p(x; \theta(u)) = \exp\{\sum_{i=1}^{m} \theta^i(u)F_i(x) + C(x) - \psi(\theta(u))\},$$

where x is a random variable, $\theta = \{\theta^1, \cdots, \theta^n\}$ is the natural coordinates or canonical parameters, u denote the local parameter and $F(x) = \{F_1(x), \cdots, F_n(x)\}$ are sufficient statistics for $\theta = \{\theta^1, \cdots, \theta^n\}$. Let $\ell(\theta(u), x) = \log p(x; \theta(u))$ be the log likelihood of the curved exponential family and $\mathcal{J}\theta$ the Jacobian matrix of the natural parameter θ as a function of local parameter u. Then,

$$\begin{aligned} \mathcal{J}(\ell(\theta(u), x)) &= \mathcal{J}(\sum_{i=1}^{n} \theta^i(u)F_i(x) - \psi(\theta(u))) \\ &= \mathcal{J}(\{\theta^T(u)F(x) - \psi(\theta(u))\}) \\ &= \mathcal{J}(\theta^T(u)\{F(x) - \eta(u)\}), \end{aligned}$$

where $\eta(u) = E[F(x)]$ and $\mathcal{J}\psi(\theta(u)) = \mathcal{J}\theta^T(u)\mathcal{J}_\theta\psi(\theta) = \mathcal{J}\theta^T(u)\eta(u)$.

The maximum likelihood estimator $\hat{u}_{ML}$ of the curved exponential family satisfies the following likelihood equation.

$$\mathcal{J}\ell(\hat{u}_{ML}) = \mathcal{J}\log p(x,u) = \mathcal{J}\theta^T(\hat{u}_{ML})[F(x) - \eta(\hat{u}_{ML})] = 0. \tag{5.14}$$

Here $\ell(\hat{u}_{ML})$ is an objective function to be maximized with parameter u. In [61], Amari suggested the use of natural gradient updates for the optimization on a Riemannian manifold. That is,

$$u^{new} = u + \lambda G^{-1}(u)\mathcal{J}\ell(u), \tag{5.15}$$

where λ is a positive learning rate that determines the step size and $G(u)$ denote the matrix of Riemannian metric of the manifold. Here G is the Fisher information matrix which can be represented as

$$G(u) = \mathcal{J}\theta^T(u)G(\theta)\mathcal{J}\theta(u),$$

where $G(\theta)$ is the Fisher information matrix with respect to the natural parameter θ. A recursive MLE of curved exponential family is obtained as follows

$$\begin{aligned} u^{(k+1)} &= u^{(k)} + \lambda G^{-1}(u^{(k)})\mathcal{J}\ell(u^{(k)}) \\ &= u^{(k)} + \lambda G^{-1}(u^{(k)})\mathcal{J}\theta^T(u^{(k)})\Big[F(x) - \eta(u^{(k)})\Big], \end{aligned} \tag{5.16}$$

where $\theta(u)$ and $\eta(u)$ are the natural parameter and the expectation parameter of the distribution, respectively and $F(x)$ is the sufficient statistics. Now, the algorithm is summarized as follows [34].

The natural gradient based iterative MLE algorithm

1. Distribution reparameterization
   $p(x;u) = p(x;\theta(u)) = \exp\{\sum_{i=1}^{m}\theta^i(u)F_i(x) + C(x) - \psi(\theta(u))\}$.
   Identify the natural parameter $\theta(u)$, sufficient statistics $F(x)$ and potential function $\psi(\theta(u))$.

2. Find expectation parameter η and Fisher information metric G for the curved exponential family $p(x;\theta(u))$.
   $\eta(u) = E[F(x)] = \mathcal{J}_\theta\psi(\theta)$.
   $G(\theta) = \mathcal{J}_\theta\mathcal{J}_\theta^T\psi(\theta)$.
   $G(u) = \mathcal{J}\theta^T(u)G(\theta)\mathcal{J}\theta(u)$.

3. Input initial conditions
   $u^0,\ \ G(u^0) = \mathcal{J}\theta^T(u^0)G(\theta(u^{(0)}))\mathcal{J}\theta(u^{(0)})$

4. Set step size $e^{(k)} > \epsilon > 0$, $k = 0$,
   while $e^{(k)} > \epsilon$
   Loop for the $(k+1)^{th}$ iteration
   $u^{(k+1)} = u^{(k)} + \lambda G^{-1}(u^{(k)})\mathcal{J}\theta^T(u^{(k)})\Big[F(x) - \eta(u^{(k)})\Big]$
   $G(u^{(k+1)}) = \mathcal{J}\theta^T(u^{(k+1)})G(\theta(u^{(k+1)}))\mathcal{J}\theta(u^{(k+1)})$
   Update step size
   $e^{(k+1)} = ||(u^{(k)} - u^{(k+1)})||$
   end

   Continue this process until it converges.

Remark 5.6. The natural gradient estimator updates the underlying manifold metric, the Fisher information matrix G at each iteration, which evaluates the estimate accuracy. For the cases where the underlying parameter spaces are not Euclidean but are curved the standard gradient does not represent the steepest descent direction in the parameter space. The natural gradient updates in the equation (5.15) improve the steepest descent update rule by taking the geometry of the Riemannian manifold into account to calculate the learning directions. In fact, it modifies the standard gradient direction according to the local curvature of the parameter space in terms of the Riemannian metric matrix $G(u)$, thus offers faster convergence than the steepest descent method. The natural gradient approach increases the stability of the iteration with respect to Newton's method through replacing the Hessian by its expected value, that is, the Riemannian metric matrix $G(u)$. In the proposed iterative MLE algorithm in equation (5.16), an alternative parametrization of the curved exponential family in terms of the natural and expectation parameters are employed. Through such a reparametrization, the implementation of the natural gradient updates is facilitated by the relevant statistics of a curved exponential family.

Now, we discuss the geometrical explanation of the convergence of the non-linear iterative estimator in terms of the information geometry [34]. In figure (5.1), let $\mathcal{A}$ be the natural parameter space and $\mathcal{B}$ be the dual space. The curved exponential family be represented by the curve $\mathcal{F}_A = \{\theta(u) : u \in \mathbb{R}^m\}$ in $\mathcal{A}$ and $\mathcal{F}_B = \{\eta(u) : u \in \mathbb{R}^m\}$ in the dual space $\mathcal{B}$. Starting from an initial parameter $u^{(k)}$, the algorithm constructs a vector $\mathcal{L}_{u^k} = \{\tilde{\eta}(u^{(k)}) = F(x) - \eta(u^k)\}$ from the current distribution represented by its expectation parameter $\eta(u^{(k)})$ to the measurement $F(x)$. The projection of $\tilde{\eta}(u^k)$ to the tangent vector $\mathcal{J}\theta(u^{(k)})$ of the natural parameter $\theta(u^k)$ with respect to the metric $G(u^{(k)})$ gives the steepest descent gradient (natural gradient) to update the current estimates (where $\mathcal{J}\theta(u^{(k)})$ is represented by the dashed arrow in both $\mathcal{A}$ and $\mathcal{B}$, while the natural gradient is represented by the solid arrow in $\mathcal{B}$).

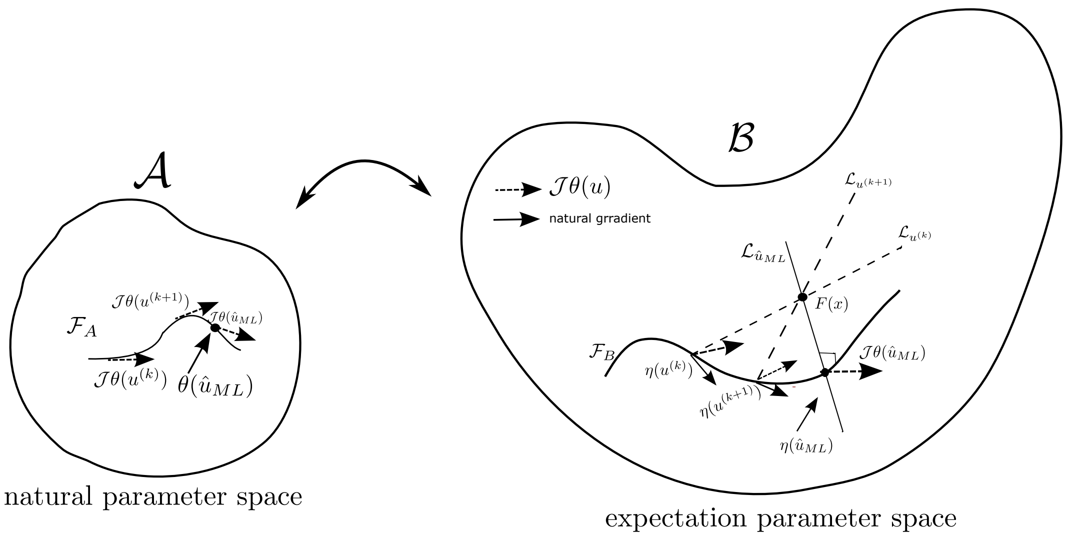

Figure 5.1: Convergence of the iterative maximum likelihood estimator algorithm [34].

The iteration continue according to the equation (5.16) until the two vectors $\tilde{\eta}(u^k)$ and $\mathcal{J}\theta(u^{(k)})$ are orthogonal to each other. Then, the algorithm achieves convergence with the steepest descent gradient $G^{-1}(u^k)\mathcal{J}\theta^T(u^{(k)})\tilde{\eta}(u^k)$ vanishes and a solution to the MLE equation (5.14) is obtained by projecting the data $F(x)$ onto $\mathcal{F}_B$ orthogonally to $\mathcal{J}\theta(u)$.

5.3 Parametrized Measure Models and Fisher-Neyman Sufficient Statistic

In this section, we discuss the Fisher-Neyman sufficient statistic for a parametrized measure model. We show that the Fisher-Neyman sufficient statistic is invariant under isostatistical immersions of statistical manifolds.

Let (Ω, Σ) be a measurable space, denote

$$\begin{aligned}
\mathcal{P}(\Omega) &= \{\mu : \mu \ is \ a \ probability \ measure \ on \ \Omega\}. \\
\mathcal{M}(\Omega) &= \{\mu : \mu \ is \ a \ finite \ measure \ on \ \Omega\}. \\
\mathcal{S}(\Omega) &= \{\mu : \mu \ is \ a \ signed \ finite \ measure \ on \ \Omega\}.
\end{aligned}$$

Clearly $\mathcal{P}(\Omega) \subset \mathcal{M}(\Omega) \subset \mathcal{S}(\Omega)$ and note that $\mathcal{S}(\Omega)$ is a real vector space. Define, for $\mu \in \mathcal{S}(\Omega)$,

$$\| \mu \|_{TV} = \sup \sum_{i=1}^{n} | \mu(A_i) |,$$

where the supremum is taken over all the finite partitions $\Omega = A_1 \cup A_2 \cup ... \cup A_n$ with disjoint sets $A_i \in \Sigma$. Then, $\mathcal{S}(\Omega)$ is a Banach space with respect to $\| \,.\, \|_{TV}$. Note that $\mathcal{P}(\Omega) = \{\mu \in \mathcal{M}(\Omega) : \| \mu \|_{TV} = 1\}$. A subset $A \subset \Omega$ is called the null set of a measure μ if $\mu(A) = 0$. Let μ_1 and μ_2 be two non-negative finite measures on Ω, then μ_1 is said to dominate μ_2 (denoted by, $\mu_2 \ll \mu_1$) if every null set of μ_1 is also a null set of μ_2.

Fix a measure $\mu_0 \in \mathcal{M}(\Omega)$, then define

$$\begin{aligned} \mathcal{P}(\Omega, \mu_0) &= \{\mu \in \mathcal{P}(\Omega) : \quad \mu \ll \mu_0\}. \\ \mathcal{M}(\Omega, \mu_0) &= \{\mu \in \mathcal{M}(\Omega) : \quad \mu \ll \mu_0\}. \\ \mathcal{S}(\Omega, \mu_0) &= \{\mu \in \mathcal{S}(\Omega) : \quad \mu \ll \mu_0\}. \end{aligned}$$

We identify $\mathcal{S}(\Omega, \mu_0)$ with $L^1(\Omega, \mu_0)$ by the canonical map

$$\begin{aligned} i : \mathcal{S}(\Omega, \mu_0) &\longrightarrow L^1(\Omega, \mu_0) \\ \mu &\rightarrow \frac{d\mu}{d\mu_0}, \end{aligned}$$

where $\frac{d\mu}{d\mu_0}$ is the Radon-Nikodym derivative of μ with respect to μ_0 ([62]). Using this identification $\mathcal{M}(\Omega, \mu_0) = \{\phi\mu_0 : \phi \geq 0\}$ and $\mathcal{P}(\Omega, \mu_0) = \{\phi\mu_0 : \int_\Omega \phi\mu_0 = 1\}$. Also, note that

$$\| \phi \|_{L^1(\Omega,\mu_0)} = \| \phi\mu_0 \|_{TV} \,.$$

Definition 5.6. Let V and W be Banach spaces and $U \subset V$ be an open subset. A map $\phi : U \longrightarrow W$ is called differentiable at $x \in U$, if there exists a bounded linear operator $d_x\phi \in L(V, W)$ such that

$$\lim_{h \longrightarrow 0} \frac{\| \phi(x + h) - \phi(x) - d_x\phi(h) \|_W}{\| h \|_V} = 0.$$

In this case, $d_x\phi$ is called the differential of ϕ at x. Moreover ϕ is called continuously differentiable or shortly a c^1-map, if it is differentiable at every $x \in U$ and the map

$d\phi : U \longrightarrow L(V, W)$ defined by $x \longrightarrow d_x\phi$ is continuous. Furthermore, a differentiable map $c : (-\varepsilon, \varepsilon) \longrightarrow W$ is called a curve in W.

Note. Let $X \subset V$ be an arbitrary subset and let $x_0 \in X$. Then $v \in V$ is called a tangent vector of X at x_0 if there is a curve $c : (-\varepsilon, \varepsilon) \longrightarrow X$ such that $c(0) = x_0$ and $\frac{dc}{dt}(0) = v$. The set of all tangent vectors at x_0 is called the tangent space of X at x_0 and is denoted by $T_{x_0}X$.

Remark 5.7. In [63], Ay et al. proved that the tangent space of $\mathcal{M}(\Omega)$ and $\mathcal{P}(\Omega)$ at μ are $\mathcal{S}(\Omega, \mu)$ and $\mathcal{S}_0(\Omega, \mu)$ respectively, where $\mathcal{S}_0(\Omega, \mu) = \{\mu \in \mathcal{S}(\Omega, \mu) : \mu(\Omega) = 0\}$.

5.3.1 Fisher-Neyman Sufficient Statistics and Isostatistical Immersions

In this subsection, we discuss the Fisher-Neyman sufficient statistic for parametrized measure models. Also, we prove invariants of Fisher-Neyman sufficient statistic under isostatistical immersions.

Definition 5.7. Let $(\Omega, \mathcal{B})$ and $(\Omega^{'}, \mathcal{B}^{'})$ be two measurable spaces. A measurable map $k : \Omega \longrightarrow \Omega^{'}$ is called a statistic. For a statistic there is an induced map $k_* : \mathcal{S}(\Omega) \longrightarrow \mathcal{S}(\Omega^{'})$ defined by $k_*\mu(A) = \mu(k^{-1}(A))$, for A in the σ-algebra $\mathcal{B}^{'}$ of $\Omega^{'}$.

Note that $k_* : \mathcal{S}(\Omega) \longrightarrow \mathcal{S}(\Omega^{'})$ is a bounded linear map which is monotone. That is, it maps non-negative measures to non-negative measures. Using Jordan decomposition theorem ([62]) of measures, $k_*(\mathcal{P}(\Omega)) \subset \mathcal{P}(\Omega^{'})$. Also, from the definition of k_* we get if $\mu_1 << \mu_2$, then $k_*\mu_1 << k_*\mu_2$. Hence, $k_* : \mathcal{S}(\Omega, \mu) \longrightarrow \mathcal{S}_0(\Omega^{'}, k_*\mu)$ is a bounded linear map.

The elements of $\mathcal{S}(\Omega, \mu)$ are of the form $\phi\mu$ for $\phi \in L^1(\Omega, \mu)$ and if we write $k_*(\phi\mu) = \phi^{'} k_*\mu$ for $\phi^{'} \in L^1(\Omega^{'}, k_*\mu)$, then $\phi^{'}$ is called the conditional expectation of ϕ given k. This gives a bounded linear map $k_*^\mu : L^1(\Omega, \mu) \longrightarrow L^1(\Omega^{'}, k_*\mu)$ defined by $k_*^\mu(\phi) = \phi^{'}$.

The pullback of a measurable function $\phi^{'} : \Omega^{'} \longrightarrow \mathbb{R}$ is defined as $k^*\phi^{'} = \phi^{'} \circ k$. If $A^{'} \subset \Omega^{'}$ and $A = k^{-1}(A^{'})$ we have $\chi_A = k^*\chi_{A^{'}}$ and thus $k_*\mu(A) = \mu(k^{-1}(A))$ is equivalent to $\chi_{A^{'}} k_*\mu = k_*(\chi_A\mu) = k_*(k^*\chi_{A^{'}}\mu)$. By the linearity it extends to the step functions on $\Omega^{'}$ and by the density of step functions in $L^1(\Omega^{'}, k_*\mu)$ we have, for any $\phi^{'} \in L^1(\Omega^{'}, k_*\mu)$

$$k_*(k^*\phi^{'}\mu) = \phi^{'} k_*\mu$$

and hence $k_*^\mu(k^*\phi^{'}) = \phi^{'}$.

Definition 5.8. Let $k : \Omega \longrightarrow \Omega^{'}$ be a statistic and $\mu^{'} \in \mathcal{M}(\Omega^{'})$. A k-congruent embedding is a bounded linear map $K_* : \mathcal{S}(\Omega^{'}, \mu^{'}) \longrightarrow \mathcal{S}(\Omega)$ such that

- K_* is monotone.
- $k_*(K_*(\nu^{'})) = \nu^{'}$ for all $\nu^{'} \in \mathcal{S}(\Omega^{'}, \mu^{'})$.

Furthermore, the image of a k-congruent embedding K_* in $\mathcal{S}(\Omega)$ is called the k-congruent subspace of $\mathcal{S}(\Omega)$.

Example 5.2. Let $k : \Omega \longrightarrow \Omega^{'}$ be a statistic and $\mu \in \mathcal{M}(\Omega)$ be with $\mu^{'} = k_*\mu \in \mathcal{M}(\Omega^{'})$. Then, the map $K_\mu : \mathcal{S}(\Omega^{'}, \mu^{'}) \longrightarrow \mathcal{S}(\Omega, \mu)$ defined by $K_\mu(\phi^{'}\mu^{'}) = k^*\phi^{'}\mu$ for $\phi^{'} \in L^1(\Omega^{'}, \mu^{'})$ is a $k-$congruent embedding, since

$$\begin{aligned} k_*(K_\mu(\phi^{'}\mu^{'})) &= k_*(k^*\phi^{'}\mu) \\ &= \phi^{'}k_*\mu \\ &= \phi^{'}\mu^{'}. \end{aligned} \tag{5.17}$$

Definition 5.9. A Markov kernel between two measurable space $(\Omega, \mathcal{B})$ and $\Omega^{'}, \mathcal{B}^{'})$ is a map $K : \Omega \longrightarrow \mathcal{P}(\Omega^{'})$ associating to each $\omega \in \Omega$ a probability measure on $\Omega^{'}$ such that for each fixed measurable $A^{'} \subset \Omega^{'}$ the map

$$\Omega \longrightarrow [0,1], \quad \omega \longrightarrow K(\omega)(A^{'}) =: K(\omega; A^{'})$$

is measurable for all $A^{'} \in \mathcal{B}^{'}$. The Markov morphism induced by K is the linear map

$$K_* : \mathcal{S}(\Omega) \longrightarrow \mathcal{S}(\Omega^{'}), \quad K_*\mu(A^{'}) = \int_\Omega K(\omega; A^{'}) d\mu(\omega).$$

Use the notation $K_*(\mu; A^{'}) = K_*\mu(A^{'})$. Since $K(\omega) \in \mathcal{P}(\Omega^{'})$, it follows that $K(\omega; \Omega^{'}) = 1$, for all $\omega \in \Omega$, hence $K_*\mu(\Omega^{'}) = \mu(\Omega)$. Thus,

$$\parallel K_*\mu \parallel_{TV} = \parallel \mu \parallel_{TV}, \quad \text{for all } \mu \in \mathcal{M}(\Omega).$$

In particular, a Markov morphism maps probability measures to probability measures. For a general measure $\mu \in \mathcal{S}(\Omega)$, we have $|K_*(\mu; A^{'})| \leq K_*(|\mu|; A^{'})$ for all $A^{'} \in \mathcal{B}^{'}$ and hence,

$$\parallel K_*\mu \parallel_{TV} \;\; \leq \;\; \parallel K_* \mid \mu \mid \parallel_{TV} = \parallel \mu \parallel_{TV}, \quad \text{for all } \mu \in \mathcal{S}(\Omega)$$

so that $K_* : \mathcal{S}(\Omega) \longrightarrow \mathcal{S}(\Omega^{'})$ is a bounded linear map.

Note that the Markov kernel K can be recovered from K_* using the relation

$$K(\omega) = K_*\delta^\omega, \text{ for } \omega \in \Omega,$$

where δ^ω denotes the Dirac measure supported at $\omega \in \Omega$.

Note. From the definition of Markov morphism it is immediate that K_* preserves the dominance of measure, that is, if $\mu_1 << \mu_2$, then $K_*\mu_1 << K_*\mu_2$. Thus, for each $\mu \in \mathcal{M}(\Omega)$ there is a restriction

$$K_* : \mathcal{S}(\Omega, \mu) \longrightarrow \mathcal{S}(\Omega^{'}, \mu^{'}),$$

where $\mu^{'} = K_*\mu$. This induces a bounded linear map

$$K_*^\mu : L^1(\Omega, \mu) \longrightarrow L^1(\Omega^{'}, \mu^{'}), \quad \phi \longrightarrow \phi^{'},$$

where $\phi^{'}$ is given by $K_*(\phi\mu) = \phi^{'}\mu^{'}$ and $\phi^{'}$ is called the conditional expectation of ϕ given K.

Definition 5.10. A Markov kernel $K : \Omega^{'} \longrightarrow \mathcal{P}(\Omega)$ is called k-congruent for the statistic $k : \Omega \longrightarrow \Omega^{'}$ if $k_*K(\omega^{'}) = \delta^{\omega^{'}}$ for all $\omega^{'} \in \Omega^{'}$.

Remark 5.8. Let $k : \Omega \longrightarrow \Omega^{'}$ be a statistic, then there is an induced Markov kernel $K^k : \Omega \longrightarrow \mathcal{P}(\Omega^{'})$ defined by $K^k(\omega) = \delta^{k(\omega)}$, so that $K^k(\omega; A^{'}) = \chi_{k^{-1}(A^{'})}(\omega)$. In this case, Markov morphism induced by K^k coincides with $k_* : \mathcal{S}(\Omega) \longrightarrow \mathcal{S}(\Omega^{'})$. Also note that if K is a k-congruent Markov kernel, then $(K^kK)_*$ is an identity map on $\mathcal{S}(\Omega^{'})$.

Definition 5.11. Let Ω be a measurable space, a parametrized measure model is a triplet $(\mathbf{M}, \Omega, \mathbf{p})$ where $\mathbf{M}$ is a Banach manifold and $\mathbf{p} : \mathbf{M} \longrightarrow \mathcal{M}(\Omega) \subset \mathcal{S}(\Omega)$ is a c^1-map.

The triplet $(\mathbf{M}, \Omega, \mathbf{p})$ is called a statistical model if the image of $\mathbf{p}$ is contained in $\mathcal{P}(\Omega)$. Such a model is said to be dominated by μ_0 if the image of $\mathbf{p}$ is contained in $\mathcal{M}(\Omega, \mu_0)$. In this case the notation used for this model is $(\mathbf{M}, \Omega, \mu_0, \mathbf{p})$.

Remark 5.9. If a parametrized measure model $(\mathbf{M}, \Omega, \mu_0, \mathbf{p})$ is dominated by μ_0, then there is a density function $p : \Omega \times \mathbf{M} \longrightarrow \mathbb{R}$ such that $\mathbf{p}(\xi) = p(.;\xi)\mu_0$. The density function p is said to be a regular density if for all $V \in T_\xi\mathbf{M}$ the partial derivative $\partial_V p(.;\xi)$ exists and lies in $L^1(\Omega, \mu_0)$.

Definition 5.12. Let $(\mathbf{M}, \Omega, \mathbf{p})$ be a parametrized measure model and $k : \Omega \longrightarrow \Omega^{'}$ be a statistic. Suppose that there is a $\mu \in \mathcal{M}(\Omega)$ such that $\mathbf{p}(\xi) = p^{'}(k(.);\xi)\mu$, for some $p^{'} \in L^1(\Omega^{'}, \mu^{'})$ and hence,

$$\mathbf{p}^{'}(\xi) = k_*\mathbf{p}(\xi) = p^{'}(.;\xi)\mu^{'}, \tag{5.18}$$

where $\mu^{'} = k_*\mu$. Then, k is called a Fisher-Neyman sufficient statistic for the model $(\mathbf{M}, \Omega, \mathbf{p})$.

Proposition 5.2. *[36] Let $(\mathbf{M}, \Omega, \mu_0, \mathbf{p})$ be a parametrized measure model given by the density function $\mathbf{p}(\xi) = p(\omega; \xi)\mu_0$ and let $k : \Omega \longrightarrow \Omega^{'}$ be a statistic with the induced parametrized measure model $(\mathbf{M}, \Omega^{'}, \mu_0^{'}, \mathbf{p}^{'})$, where $\mathbf{p}^{'}(\xi) = k_*\mathbf{P}(\xi)$ and $\mu_0^{'} = k_*\mu_0$, given by $\mathbf{p}^{'}(\xi) = p^{'}(\omega^{'}; \xi)\mu_0^{'}$. Then, k is a Fisher-Neyman sufficient statistic for $(\mathbf{M}, \Omega, \mu_0, \mathbf{p})$ if and only if there is a function $r \in L^1(\Omega, \mu_0)$ such that $p, p^{'}$ can be chosen such that*

$$p(\omega; \xi) = r(\omega)p^{'}\left(k(\omega); \xi\right). \tag{5.19}$$

Proof. Let $\xi \in \mathbf{M}$ be arbitrary, define

$$A_\xi = \{\omega \in \Omega : p(\omega; \xi) = 0\} \quad \text{and} \quad B_\xi = \{\omega \in \Omega : p^{'}(k(\omega); \xi) = 0\}.$$

Then $\mathbf{P}(\xi)(B_\xi) \leq \mathbf{P}(\xi)(k^{-1}k(B_\xi)) = \mathbf{P}^{'}(\xi)(k(B_\xi)) = 0$, last equality follows from the definition of B_ξ. Thus, $\mathbf{P}(\xi)(B_\xi) = 0$ and hence, $B_\xi \setminus A_\xi$ in Ω is a null set with respect to μ_0. Set $p(\omega; \xi) \equiv 0$ on $B_\xi \setminus A_\xi$, so that we assume that $B_\xi \setminus A_\xi$ is empty. That is, $B_\xi \subset A_\xi$. Use the convention $\frac{0}{0} =: 1$ to define the measurable function

$$r(\omega; \xi) = \frac{p(\omega; \xi)}{p^{'}(k(\omega); \xi)}.$$

Now, it is enough to show that r is independent of ξ and has a finite integral if and only if k is a Fisher-Neyman sufficient statistic.
Assume that k is Fisher-Neyman sufficient statistic. Then, there is a measure $\mu \in \mathcal{M}(\Omega, \mu_0)$ such that $\mathbf{p}(\xi) = \tilde{p}^{'}(k(\omega); \xi)\mu$ for some $\tilde{p}^{'} : \Omega^{'} \times \mathbf{M} \longrightarrow \mathbb{R}$ with $\tilde{p}^{'}(.; \xi) \in L^1(\Omega^{'}, \mu^{'})$ for all ξ. Then,

$$\mathbf{P}^{'}(\xi) = k_*\mathbf{P}(\xi) = k_*\left(k^*(\tilde{p}^{'}(.; \xi))\mu\right) = \tilde{p}^{'}(.; \xi)\mu^{'}$$

by (5.18). Since $\mathbf{P}(\xi)$ is dominated by both μ and μ_0, assume without loss of generality that μ is dominated by μ_0 and hence, $\mu^{'}$ is dominated by $\mu_0^{'}$. Denote

$$\psi_0 = \frac{d\mu}{d\mu_0} \in L^1(\Omega, \mu_0) \quad \text{and} \quad \psi_0^{'} = \frac{d\mu^{'}}{d\mu_0^{'}} \in L^1(\Omega^{'}, \mu_0^{'}).$$

Then

$$p(\omega;\xi) = \frac{d\mathbf{P}(\xi)}{d\mu_0} = \frac{d\mathbf{P}(\xi)}{d\mu}\frac{d\mu}{d\mu_0}(\omega;\xi) = \tilde{p}'(k(\omega);\xi)\psi_0(\omega),$$

and similarly,

$$p'(\omega',\xi) = \tilde{p}'(\omega';\xi)\psi_0'(\omega').$$

Therefore,

$$r(\omega;\xi) = \frac{p(\omega;\xi)}{p'(k(\omega);\xi)} = \frac{\psi_0(\omega)}{\psi_0'(k(\omega))}.$$

This implies r is independent of ξ. Now

$$\begin{aligned}\int_\Omega r(\omega)d\mu_0 &= \int_\Omega \frac{\psi_0(\omega)}{\psi_0'(k(\omega))}d\mu_0 = \int_\Omega k^*\left(\frac{1}{\psi_0'(\omega')}\right)d\mu \\ &= \int_{\Omega'} \frac{1}{\psi_0'(\omega')}d\mu' = \int_{\Omega'} d\mu_0' = \mu_0(\Omega) < \infty.\end{aligned}$$

So $r \in L^1(\Omega,\mu_0)$.
Conversely, assume that equation (5.19) holds and $r \in L^1(\Omega,\mu_0)$ then for $\mu = r\mu_0 \in \mathcal{M}(\Omega,\mu_0)$

$$\mathbf{P}(\xi) = p(.;\xi)\mu_0 = p'(k(.);\xi)\mu. \tag{5.20}$$

Hence the proposition follows from the definition of the Fisher-Neyman sufficient statistic.

□

Theorem 5.4. *[36] Let $(\mathbf{M},\Omega,\mu_0,\mathbf{p})$ be a parametrized measure model given by the regular density function $\mathbf{p}(\xi) = p(\omega;\xi)\mu_0$ and let $k:\Omega \longrightarrow \Omega'$ be a statistic. Then, k is a Fisher-Neyman sufficient statistic for the parametrized measure model if and only if there exists a function $s:\Omega'\times\mathbf{M}\longrightarrow\mathbb{R}$ and a function $t \in L^1(\Omega,\mu_0)$ such that for all $\xi \in \mathbf{M}$ we have $s(\omega',\xi) \in L^1(\Omega',k_*\mu_0)$ and*

$$p(\omega;\xi) = s(k(\omega);\xi)t(\omega) \tag{5.21}$$

Proof. If (5.21) holds, let $\mu = t(\omega)\mu_0 \in \mathcal{M}(\Omega,\mu_0)$. Then for all ξ

$$\mathbf{P}(\xi) = p(.;\xi)\mu_0 = s(k(\omega);\xi)\mu, \tag{5.22}$$

hence k is a Fisher-Neyman sufficient statistic for $\mathbf{P}$.

Conversely, if k is a Fisher-Neyman sufficient statistic for $\mathbf{P}$, take $t = r$ and $s = p'$ in (5.19. Then (5.21) follows. $\square$

Let $(\mathbf{M}, g, C)$ be a statistical manifold. A smooth family of probability distributions $p(x;\xi)$ on a sample space Ω with parameter $\xi \in \mathbf{M}$ is called a probability density for g and C if

$$\begin{aligned} g(\xi; V_1, V_2) &= \mathbf{E}_\xi \left(\frac{\partial}{\partial V_1} \log p(.;\xi) \frac{\partial}{\partial V_2} \log p(.;\xi) \right) \\ C(\xi; V_1, V_2, V_3) &= \mathbf{E}_\xi \left(\frac{\partial}{\partial V_1} \log p(.;\xi) \frac{\partial}{\partial V_2} \log p(.;\xi) \frac{\partial}{\partial V_3} \log p(.;\xi) \right), \end{aligned}$$

where $V_i \in T_\xi \mathbf{M}$ for $i = 1, 2, 3$.

Definition 5.13. A smooth map f from a statistical manifold $(\mathbf{M}_1, g_1, C_1)$ to a statistical manifold $(\mathbf{M}_2, g_2, C_2)$ is said to be an isostatistical immersion if f is an immersion of $\mathbf{M}_1$ into $\mathbf{M}_2$ such that $g_1 = f^*(g_2)$, $C_1 = f^*(C_2)$.

Note. Let $f : (\mathbf{M_1}, \nabla, g) \longrightarrow (\mathbf{M_2}, \tilde{\nabla}, \tilde{g})$ be a statistical immersion and C_1, C_2 be the cubic forms on $\mathbf{M_1}$ and $\mathbf{M_2}$, respectively. Then,

$$\begin{aligned} C_1(X, Y, Z) &= (\nabla_X g)(Y, Z) \\ &= Xg(Y, Z) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) \\ &= f_* X \tilde{g}(f_* Y, f_* Z) - g(\tilde{\nabla}_{f_* X} f_* Y, f_* Z) - g(f_* Y, \tilde{\nabla}_{f_* X} f_* Z) \\ &= C_2(f_* X, f_* Y, f_* Z). \end{aligned}$$

Hence f is an isostatistical immersion. Conversely if f is an isostatistical immersion, then f is a statistical immersion for the statistical structures induced from cubic forms. Thus, the isostatistical immersion defined above and the statistical immersion (cf. Definition(2.7)) are the same.

Lemma 5.1. *[36] Assume that $f : (\mathbf{M}_1, g_1, C_1) \longrightarrow (\mathbf{M}_2, g_2, C_2)$ is an isostatistical immersion. If there exists a measure space Ω and a function $p(\omega; \xi_2) : \Omega \times \mathbf{M}_2 \longrightarrow \mathbb{R}$ such that p is a probability density for the tensors g_2 and C_2, then $f^*(p)(\omega; \xi_1) = p(\omega; f(\xi_1))$ is a probability density for g_1 and C_1.*

Proof. Since f is an isostatistical immersion we have,

$$\begin{aligned} g_1(\xi; V_1, V_2) &= g_2(h(\xi); f_*(V_1), f_*(V_2)), \\ &= \int_\Omega \frac{\partial}{\partial f_* V_1} \log(p(x; f(\xi))) \frac{\partial}{\partial f_* V_2} \log(p(x; f(\xi))) p(x; f(\xi)) dx, \\ &= \mathbf{E}_{f^*(p)} \left(\frac{\partial}{\partial V_1} \log f^*(p)(.;\xi) \frac{\partial}{\partial V_2} \log f^*(p)(.;\xi) \right). \end{aligned}$$

This implies $f^*(p)$ is a probability density for g_1. Similarly we can prove that $f^*(p)$ is a probability density for C_1. □

Let $f : (\mathbf{M}_1, g_1, C_1) \longrightarrow (\mathbf{M}_2, g_2, C_2)$ be an isostatistical immersion with parametrized model $(\mathbf{M}_2, \Omega, \mu, \mathbf{p})$. That is, $\mathbf{p}(\xi_2) = p(.;\xi_2)\mu$ where $p(\omega; \xi_2) : \Omega \times \mathbf{M}_2 \longrightarrow \mathbb{R}$ is the density function. Then, there is an induced parametrized model given by f, denoted by $(\mathbf{M}_1, \Omega, \mu, \overline{\mathbf{p}})$, where $\overline{\mathbf{p}}$ is given by $\overline{\mathbf{p}} : \mathbf{M}_1 \longrightarrow \mathcal{P}(\Omega)$ defined as $\overline{\mathbf{p}} = \mathbf{p} \circ f$. That is, $\overline{\mathbf{p}}(\xi_1) = \mathbf{p}(f(\xi_1)) = p(\omega; f(\xi_1))\mu$. Define $f^*(p)(\omega; \xi_1) = p(\omega; f(\xi_1))$, then $f^*(p)(\omega; \xi_1) : \Omega \times \mathbf{M}_1 \longrightarrow \mathbb{R}$ is the density corresponds to the induced model.

Now, let $k : \Omega \longrightarrow \Omega'$ be a statistic then we have two models induced by k and they are $(\mathbf{M}_2, \Omega', \mu', \mathbf{p}')$ and $(\mathbf{M}_1, \Omega', \mu', \overline{\mathbf{p}}')$, where $\mu' = k_*\mu$, $\mathbf{p}'(\xi_2) = k_*\mathbf{p}(\xi_2)$ and $\overline{\mathbf{p}}' = \mathbf{p}' \circ f$. Also, if $\mathbf{p}'(\xi_2) = p'(\omega; \xi_2)\mu'$, then $\overline{\mathbf{p}}'(\xi_1) = f^*(p')(\omega; \xi_1)\mu'$. Now, we have the following theorem.

Theorem 5.5. *Let $k : \Omega \longrightarrow \Omega'$ be a statistic and $f : (\mathbf{M}_1, g_1, C_1) \longrightarrow (\mathbf{M}_2, g_2, C_2)$ be an isostatistical immersion with parametrized model $(\mathbf{M}_2, \Omega, \mu, \mathbf{p})$. Suppose k is a Fisher-Neyman sufficient statistic with respect to $(\mathbf{M}_2, \Omega, \mu, \mathbf{p})$ and $(\mathbf{M}_2, \Omega', \mu', \mathbf{p}')$. Then, k is a Fisher-Neyman sufficient statistic with respect to $(\mathbf{M}_1, \Omega, \mu, \overline{\mathbf{p}})$ and $(\mathbf{M}_1, \Omega', \mu', \overline{\mathbf{p}}')$.*

Proof. Given that k is a Fisher-Neyman sufficient statistic with respect to $(\mathbf{M}_2, \Omega, \mu, \mathbf{p})$ and $(\mathbf{M}_2, \Omega', \mu', \mathbf{p}')$. By the definition of the Fisher-Neyman sufficient statistic

$$p(\omega; \xi_2) = p'(k(\omega); \xi_2).$$

This implies

$$\begin{aligned} f^*(p)(\omega; \xi_1) &= p(\omega; f(\xi_1)) \\ &= p'(k(\omega); f(\xi_1)) \\ &= f^*(p')(k(\omega); \xi_1). \end{aligned}$$

Hence, k is a Fisher-Neyman sufficient statistic with respect to $(\mathbf{M}_1, \Omega, \mu, \overline{\mathbf{p}})$ and $(\mathbf{M}_1, \Omega', \mu', \overline{\mathbf{p}}')$.

$\square$

5.4 Concluding Remarks

Here we first summarize certain significant results that we have obtained and then give a list of topics to be explored further. In the case of immersions into statistical manifolds a necessary and sufficient condition for the inherited statistical manifold structures to be dual to each other is obtained. Then, statistical immersion is defined and proved a necessary condition for a statistical manifold to be a statistical hypersurface. Its converse is also proved. A necessary and sufficient condition for a statistical immersion into a dually flat statistical manifold of codimension one to be minimal is given. Also, a necessary condition is obtained for minimal statistical immersion of statistical manifolds equipped with α-connections. A necessary and sufficient condition for the inherited statistical manifold structures to be dual to each other is proved for a centro-affine immersion of codimension two into a dually flat statistical manifold. Then proved that the inherited statistical manifold structure is conformally-projectively flat in this case. We introduced the concept of a conformal submersion with horizontal distribution for Riemannian manifolds, which is a generalization of the affine submersion with horizontal distribution. A necessary condition for the existence of such a map is proved. Then compares the geodesics for a conformal submersion with horizontal distribution. A necessary and sufficient condition for the horizontal lift of a geodesic to be geodesic is obtained. In the case of conformal submersion with horizontal distribution, proved a necessary and sufficient condition for $(\mathbf{M}, \nabla, g_m)$ to become a statistical manifold.

We obtained a necessary and sufficient condition for $\mathbf{TM}$ to be a statistical manifold with the complete lift connection and the Sasaki lift metric. Then, proved a necessary and sufficient condition for the harmonicity of the identity map for conformally-projectively equivalent statistical manifolds. The conformal statistical submersion is defined which is a generalization of the statistical submersion and proved that harmonicity and conformality cannot coexist. Then, given a necessary condition for the harmonicity of the tangent map with respect to the complete lift structure on the tangent bundles. Also, proved a necessary and sufficient condition for the tangential map to be a statistical submersion. We show that if all ∇^1-autoparallel proper submanifolds of a ± 1-flat statistical manifold $\mathbf{M}$ are exponential then $\mathbf{M}$ is an exponential family. Then, proved that if submanifold of a statistical model is an exponential family, then it is a ∇^1-autoparallel submanifold. Also, we obtained that the Fisher-Neyman sufficient statistic is invariant under the isostatistical immersions of statistical manifolds.

We would like to continue our study in the following topics:

- Geometry of immersions into statistical manifolds in the general codimension case.
- Statistical manifold structures on the cotangent bundles of statistical manifolds [64].
- Harmonic maps between cotangent bundles.
- Estimation theory in the context of submersions.

Bibliography

[1] S. I. Amari and H. Nagaoka, *Method of Information Geometry. AMS Monograph*. Oxford University Press Oxford, 2000.

[2] K. Nomizu and T. Sasaki, *Affine differential geometry: geometry of affine immersions*. Cambridge University Press, 1994.

[3] S. L. Lauritzen, *Statistical manifolds*. JSTOR, 1984.

[4] T. Kurose, "Dual connections and affine geometry," *Mathematische Zeitschrift*, vol. 203, no. 1, pp. 115–121, 1990.

[5] T. Kurose, "On the divergences of 1-conformally flat statistical manifolds," *Tohoku Mathematical Journal, Second Series*, vol. 46, no. 3, pp. 427–433, 1994.

[6] F. Dillen, K. Nomizu, and L. Vranken, "Conjugate connections and Radon's theorem in affine differential geometry," *Monatshefte für Mathematik*, vol. 109, no. 3, pp. 221–235, 1990.

[7] P. W. Vos, "Fundamental equations for statistical submanifolds with applications to the Bartlett correction," *Annals of the Institute of Statistical Mathematics*, vol. 41, no. 3, pp. 429–450, 1989.

[8] K. Uohashi, A. Ohara, and T. Fujii, "1-conformally flat statistical submanifolds," *Osaka journal of mathematics*, vol. 37, pp. 501–507, 2000.

[9] T. V. Mahesh and K. S. Subrahamanian Moosath, "Immersions into statistical manifolds," *Proceedings of the National Academy of Sciences, India Section A: Physical Sciences*, pp. 1–6, 2021.

[10] H. Furuhata, "Codazzi structures induced by minimal affine immersions," *Banach Center Publications*, vol. 57, pp. 17–19, 2002.

[11] T. V. Mahesh and K. S. Subrahamanian Moosath, “Minimal statistical immersions of statistical manifolds.” *Ganita*, vol. 69, no. 2, pp. 77–87, 2019.

[12] R. Walter, “Centroaffine differential geometry: submanifolds of codimension 2,” *Results in Mathematics*, vol. 13, no. 3-4, pp. 386–402, 1988.

[13] K. Nomizu and T. Sasaki, “Centroaffine immersions of codimension two and projective hypersurface theory,” *Nagoya Mathematical Journal*, vol. 132, pp. 63–90, 1993.

[14] H. Matsuzoe, “On realization of conformally-projectively flat statistical manifolds and the divergences,” *Hokkaido mathematical journal*, vol. 27, no. 2, pp. 409–421, 1998.

[15] N. Koike and K. Takekuma, “Equiaffine immersions of general codimension and its transversal volume element map,” *Results in Mathematics*, vol. 39, no. 3-4, pp. 274–291, 2001.

[16] H. Matsuzoe, J.-I. Takeuchi, and S. I. Amari, “Equiaffine structures on statistical manifolds and bayesian statistics,” *Differential Geometry and Its Applications*, vol. 24, no. 6, pp. 567–578, 2006.

[17] O. Barndorff-Nielsen and P. Jupp, “Differential geometry, profile likelihood, L-sufficiency and composite transformation models,” *The Annals of Statistics*, pp. 1009–1043, 1988.

[18] B. O’Neill, “The fundamental equations of a submersion.” *Michigan Mathematical Journal*, vol. 13, no. 4, pp. 459–469, 1966.

[19] B. O’ Neill, *Semi-Riemannian Geometry with Application to Relativity*. Academic Press, 1983.

[20] B. O’Neill, “Submersions and geodesics,” *Duke Mathematical Journal*, vol. 34, no. 2, pp. 363–373, 1967.

[21] N. Abe and K. Hasegawa, “An affine submersion with horizontal distribution and its applications,” *Differential Geometry and its Applications*, vol. 14, no. 3, pp. 235–250, 2001.

[22] T. V. Mahesh and K. S. Subrahamanian Moosath, “Affine and conformal submersions with horizontal distribution and statistical manifolds.” *Balkan Journal of Geometry and Its Applications*, vol. 26, no. 1, pp. 34–45, 2021.

[23] H. Matsuzoe and J. Inoguchi, "Statistical structures on tangent bundles," *Appl. Sci*, vol. 5, pp. 55–75, 2003.

[24] V. Balan, E. Peyghan, and E. Sharahi, "Statistical structures on the tangent bundle of a statistical manifold with Sasaki metric," *Hacettepe Journal of Mathematics and Statistics*, vol. 49, no. 1, pp. 120–135, 2020.

[25] A. M. Pastore, M. Falcitelli, and S. Ianus, *Riemannian submersions and related topics*. World Scientific, 2004.

[26] K. Uohashi, "Harmonic maps relative to α-connections," in *Geometric Theory of Information*. Springer, 2014, pp. 81–96.

[27] F. M. ŞİMŞİR, "A note on harmonic maps of statistical manifolds," *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics*, vol. 68, no. 2, pp. 1370–1376, 2019.

[28] T. V. Mahesh and K. S. Subrahamanian Moosath, "Harmonicity of conformally-projectively equivalent statistical manifolds and conformal statistical submersions." in *GSI 2021, Lecture Notes in Computer Sciences*, vol. 12829. Springer, 2021, pp. 397–404.

[29] H. El Hendi and L. Belarbi, "Naturally harmonic maps between tangent bundles," *Balkan Journal of Geometry and Its Applications*, vol. 25, no. 1, pp. 34–46, 2020.

[30] V. Oproiu, "Harmonic maps between tangent bundles," *Rendiconti del seminario mathematico, universitá e Politecnico di Torino*, vol. 47, pp. 47–55, 1989.

[31] S. I. Amari, "Differential-geometrical methods in statistics," *Lecture Notes on Statistics*, vol. 28, p. 1, 1985.

[32] T. V. Mahesh and K. S. Subrahamanian Moosath, "Submanifolds of exponential families," *Global Journal of Advanced Research on Classical and Modern Geometries*, vol. 8, pp. 18–25, 2019.

[33] S. I. Amari, "Differential geometry of curved exponential families-curvatures and information loss," *The Annals of Statistics*, pp. 357–385, 1982.

[34] Y. Cheng, X. Wang, and B. Moran, "Optimal nonlinear estimation in statistical manifolds with application to sensor network localization," *Entropy*, vol. 19, no. 7, p. 308, 2017.

[35] G. Pistone and C. Sempi, “An infinite-dimensional structure on the space of all the probability measures equivalent to a given one,” *The Annals of Statistics*, vol. 23, pp. 1543–1561, 1995.

[36] A. Nihat, J. Jurgen, L. Hong, Van, and S. Lorenz, *Information geometry*. Springer, 2017.

[37] J. M. Lee, “Smooth manifolds,” in *Introduction to Smooth Manifolds*. Springer, 2013, pp. 1–31.

[38] S. I. Amari, *Information Geometry and Its Applications*. Springer, 2016.

[39] M. Murray and J. Rice, “Differential geometry and statistics. number 48 in monographs on statistics and applied probability,” 1993.

[40] S. Eguchi, “Second order efficiency of minimum contrast estimators in a curved exponential family,” *Annals of Statistics*, vol. 11, pp. 793–803, 1983.

[41] T. Matumoto, “Any statistical manifold has a contrast function-On the C3-functions taking the minimum at the diagonal of the product manifold,” *Hiroshima Math. J*, vol. 23, no. 2, pp. 327–332, 1993.

[42] W. Blaschke, *Vorlesungen uber Differential geometrie II*. Springer, Berlin, 1923.

[43] K. Nomizu and U. Pinkall, “On the geometry of affine immersions,” *Mathematische Zeitschrift*, vol. 195, no. 2, pp. 165–178, 1987.

[44] F. Dillen, “Equivalence theorems in affine differential geometry,” *Geometriae Dedicata*, vol. 32, no. 1, pp. 81–92, 1989.

[45] J. Zhang, “A note on curvature of α-connections of a statistical manifold,” *Annals of the Institute of Statistical Mathematics*, vol. 59, pp. 161–170, 2007.

[46] L. Ornea and G. Romani, “The fundamental equations of conformal submersions,” *Beitrage Algebra Geom*, vol. 34, no. 2, pp. 233–243, 1993.

[47] S. Gudmundsson, “On the geometry of harmonic morphisms,” *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 108, pp. 461–466, 1990.

[48] B. Fuglede, “Harmonic morphisms between riemannian manifolds,” *Annales de l’institut Fourier*, vol. 28, no. 2, pp. 107–144, 1978.

[49] T. Ishihara, "A mapping of riemannian manifolds which preserves harmonic functions," *J.Math Kyoto University*, vol. 19, pp. 215–229, 1979.

[50] S. Kobayashi and K. Nomizu, *Foundations of differential geometry. Vol I,II.* Interscience Publishers, New-York, (1963),(1969).

[51] J. Eells and J. H. Sampson, "Harmonic mappings of Riemannian manifolds," *American journal of mathematics*, vol. 86, no. 1, pp. 109–160, 1964.

[52] K. Yano and S. Kobayashi, "Prolongations of tensor fields and connections to tangent bundles *I* general theory," *Journal of the Mathematical Society of Japan*, vol. 18, no. 2, pp. 194–210, 1966.

[53] K. Yano and S. Kobayashi, "Prolongations of tensor fields and connections to tangent bundles *II* infinitesimal automorphisms," *Journal of the Mathematical Society of Japan*, vol. 18, no. 3, pp. 236–246, 1966.

[54] K. Yano and S. Kobayashi, "Prolongations of tensor fields and connections to tangent bundles *III*," *Journal of the Mathematical Society of Japan*, vol. 19, no. 4, pp. 486–488, 1967.

[55] K. Yano and S. Ishihara, *Tangent and cotangent bundles: Differential geometry*. Dekker, 1973, vol. 16.

[56] P. Dombrowski, "On the geometry of the tangent bundle." *Journal für die reine und angewandte Mathematik*, 1962.

[57] J. Eells and L. Lemaire, *Selected topics in harmonic maps*. American Mathematical Soc., 1983, vol. 50.

[58] A. Sanini, "Applicazioni armoniche tra i fibrati tangcnti di varieta riemanniane," *Bollettino dell'unione mathematica Italiana*, vol. 6, no. 2A, pp. 55–63, 1983.

[59] A. Sanini, "Applicazioni armoniche tra i fibrati tangcnti di varieta riemanniane," *Rendiconti del seminario mathematico, universitá e Politecnico di Torino*, vol. 43, no. 1, pp. 158–170, 1985.

[60] S. I. Amari, "Differential geometrical theory of statistics," *Chapter II IMS Lecture Notes*, pp. 19–94, 1987.

[61] S. I. Amari and S. C. Douglas, "Why natural gradient?" *Proceedings of the 1998 IEEE International Conference on Acoustics, Speech and Signal Processing, ICASSP'98 (Cat. No. 98CH36181)*, vol. 2, pp. 1213–1216, 1998.

[62] W. Rudin, *Real and complex analysis*. McGraw-HiII Book Company, 1987.

[63] A. Nihat, J. Jurgen, L. Hong, Van, and S. Lorenz, "Parametrized measure models," *Bernoulli*, vol. 24, no. 2, pp. 1692–1725, 2018.

[64] A. M. Blaga and A. Nannicini, "Generalized quasi-statistical structures," *Bulletin of the Belgian Mathematical Society-Simon Stevin*, vol. 27, no. 5, pp. 731–754, 2020.

List of Publications

Papers in Refereed International Journals

1. T. V. Mahesh and K. S. Subrahamanian Moosath, "Affine and conformal submersions with horizontal distribution and statistical manifolds." Balkan Journal of Geometry and Its Applications, vol. 26, no. 1, pp. 34-45, 2021.

2. T. V. Mahesh and K. S. Subrahamanian Moosath, "Immersions into Statistical Manifolds." Proceedings of the National Academy of Sciences, India Section A: Physical Sciences, Springer, pp. 1-6, 2021.

3. T. V. Mahesh and K. S. Subrahamanian Moosath, "Harmonicity of conformally-projectively equivalent statistical manifolds and conformal statistical submersions." GSI 2021 Lecture Notes in Computer Science, Springer Vol. 12829,pp. 397-404, 2021.

4. T. V. Mahesh and K. S. Subrahamanian Moosath, "Submanifolds of Exponential Families." Global Journal of Advanced Research on Classical and Modern Geometries, Vol.8, Issue 1, pp.18-25, 2019.

Refereed Conference Papers

1. T. V. Mahesh, K. S. Subrahamanian Moosath, "Harmonicity of conformally projectively equivalent statistical manifolds and conformal statistical submersions." 5^{th} International Conference on Geometric Science of Information (**GSI 21**) in PARIS, Sorbonne University, 21 - 23 July 2021.

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3. T. V. Mahesh, K. S. Subrahamanian Moosath, "Minimal statistical immersions of statistical manifolds", 67th Annual Conference of Bharat Ganita Parishad: Conference on Modern Analysis and Applications an International meet, Babu Banarasi Das University, Lucknow, 16-17 November 2019. Published in Ganita, vol 69, no. 2, 77-87, 2019.

4. T. V. Mahesh, K. S. Subrahamanian Moosath, "Information geometry - geometry of exponential family", International Conference on Differential Geometry, Algebra and Analysis, Department of Mathematics, Jamia Millia Islamia, New Delhi, 2016.

Communicated papers

1. T. V. Mahesh and K. S. Subrahamanian Moosath, "Conformal submersion with horizontal distribution".

2. T. V. Mahesh and K. S. Subrahamanian Moosath, "Harmonic Maps between Tangent Bundles of Statistical Manifolds".