CONTROLLABILITY AND OBSERVABILITY OF NETWORKED SYSTEMS

A thesis submitted

in partial fulfillment for the award of the degree of

Doctor of Philosophy

in

Mathematics

by

ABHIJITH AJAYAKUMAR

supervised by

PROF. RAJU K. GEORGE

Department of Mathematics
Indian Institute of Space Science and Technology
Thiruvananthapuram, India

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Certificate

This is to certify that the thesis titled ***CONTROLLABILITY AND OBSERVABILITY OF NETWORKED SYSTEMS*** submitted by **ABHIJITH AJAYAKUMAR**, to the Indian Institute of Space Science and Technology, Thiruvananthapuram, in partial fulfillment for the award of the degree of **Doctor of Philosophy** is a bonafide record of the original work carried out by him under my supervision. The contents of this thesis, in full or in parts, have not been submitted to any other Institute or University for the award of any degree or diploma.

PROF. RAJU K. GEORGE
Research Supervisor
Outstanding Professor, Dept. of Mathematics
Dean, Student Activities, Student Welfare & Outreach

Dr. C.V. ANILKUMAR
Professor and Head
Dept. of Mathematics

Place: Thiruvananthapuram
Date: August 2024

Declaration

I declare that this thesis titled ***CONTROLLABILITY AND OBSERVABILITY OF NETWORKED SYSTEMS*** submitted in partial fulfillment for the award of the degree of **Doctor of Philosophy** is a record of the original work carried out by me under the supervision of **PROF. RAJU K. GEORGE**, and has not formed the basis for the award of any degree, diploma, associateship, fellowship, or other titles in this or any other Institution or University of higher learning. In keeping with the ethical practice in reporting scientific information, due acknowledgments have been made wherever the findings of others have been cited.

Place: Thiruvananthapuram ABHIJITH AJAYAKUMAR

Date: August 2024 (SC19D010)

This thesis is dedicated to Teachers, Family and Friends

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Abstract

'Controllability' is a fundamental feature of dynamical systems introduced by R.E. Kalman in the 1960s. Various notions of controllability, such as state controllability, structural controllability, and so on, are proposed in the literature, and controllability conditions for both linear and non-linear systems were obtained by many authors. State controllability deals with the ability of the system to steer itself from an arbitrary initial state to a desired final state using suitable control functions, whereas structural controllability, introduced by C.T. Lin aims at setting some values to the non-zero parameters in the system matrices so that the resultant system is state controllable in the sense of Kalman. Another important feature of a control system introduced by Kalman is Observability, which focusses on the ability of reconstructing the internal states of the system from the knowledge of its outputs during an interval. Over the last few decades, investigations on controllability and observability of dynamical systems have drawn the interest of many scholars, who have made tremendous progress and acquired many new insights. Majority of these discoveries pertain to single higher-dimensional control systems. However, the prevalence of networked control systems in the actual world is far higher than that of single stand-alone control systems. In general, modelling complex systems necessitates the use of a group of separate systems linked together via an interconnection structure. The controllability and observability of large-scale complex networked systems presents fascinating research problems. There is a lot of interest in the study of controllability and observability of networked systems, since it has applications in many different scientific and technological disciplines. These studies include a range of system characteristics, including structural complexity, node dynamics and interactions between distinct nodes. Despite significant research in this area, there is no general result about the controllability and observability of networked systems in the literature that shows how the intrinsic features of the network and the dynamics of the individual nodes affect the controllability and observability of the networked system. The majority of the available results in the literature are for networked systems having identical individual nodes. However, in practice, not all individual nodes may possess the same dynamics. The objective of this thesis is to investigate the controllability and observability of networked systems having non-identical individual nodes with a focus on the effects of individual node dynamics and network topology. The obtained theoretical results are substantiated with numerical examples.

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Abbreviations

LTI	Linear Time Invariant
LTV	Linear Time Variant
NCS	Networked Control Systems
SISO	Single Input Single Output
MIMO	Multiple Input Multiple Output
PBH	Popov - Belevitch - Hautus

Nomenclature

$\mathbb{R}$	Set of all Real numbers
$\mathbb{C}$	Set of all Complex numbers
$\mathbb{R}^n$	Euclidean $n-$space
$\mathbb{K}^{m\times n}$	Set of all $m \times n$ matrices with entries from the field $\mathbb{K}$
$\mathcal{L}\left([t_0, t_f]; \mathbb{R}^m\right)$	$\left\{f \mid f : [t_0, t_f] \to \mathbb{R}^m,\ \int_{t_0}^{t_f} \lVert f(t)\rVert_{\mathbb{R}^m}^2 \, dt < \infty\right\}$
A	State matrix of the stand alone system
B	Control input matrix of the stand alone system
C	Output matrix of the stand alone system
(A, B)	Stand alone system with state matrix A and control input matrix B
$\mathcal{Q}(A, B)$	Kalman's controllability matrix
$\mathcal{O}(C, A)$	Kalman's observability matrix
$\mathcal{W}(t_0, t_f)$	Controllability Gramian matrix
$\mathfrak{M}(t_0, t_f)$	Observability Gramian matrix
$\Phi(t, t_0)$	State transition matrix
A_i	State matrix of the i^{th} node of the networked system
B_i	Control input matrix of the i^{th} node of the networked system
C_i	Output matrix of the of the i^{th} node of the networked system
D	External control input channel matrix
H_i	Inner coupling matrix of the i^{th} node of the networked system
L	Network topology matrix
Ω	State matrix of the networked system
Ψ	Control input matrix of the networked system
$diag\{a_1, a_2, \ldots, a_n\}$	$n \times n$ diagonal matrix with a_i as i^{th} diagonal entry
$uppertriang\{a_1, a_2, \ldots, a_n\}$	$n \times n$ upper-triangular matrix with a_i as i^{th} diagonal entry
$blockdiag\{A_1, A_2, \ldots, A_n\}$	block diagonal matrix with the matrix A_i as i^{th} diagonal block

Chapter 1

Introduction

The mathematical theory of control encompasses a diverse and intricate set of principles that seeks to understand and manipulate the behavior of dynamic systems. Control theory, which has its roots in linear algebra, differential equations, and optimization, offers a foundation for creating systems that display desired characteristics or responses favorably to external inputs. With its broad reach and impact, control theory has several applications in the engineering and technological domains, encompassing a variety of fields and industries. Our goal in this thesis is to derive easily verifiable controllability conditions and to investigate how different system attributes and network connections affect the controllability of networked systems.

1.1 Stand-Alone Control Systems

Systems that function independently without direct influence by external factors are called stand-alone control systems. In other words, it functions autonomously and is self-contained, capable of carrying out its control tasks without relying on coordination with other devices. These systems are made to run independently, using internal feedback and control mechanisms to make decisions and regulate processes. Stand-alone control systems are commonly employed in various applications, including industrial automation, robotics, and consumer electronics. In control theory, controllability is a fundamental concept when dealing with autonomous dynamical systems as it comes in many applications. Controllability in the context of dynamic systems — which include a wide variety of mechanical, electrical, and biological processes refers to the ability of the system to be directed from any starting state to a desired state using external inputs. To guarantee the flexibility, responsiveness, and efficiency of stand-alone dynamic systems, it is crucial to comprehend and maximize their controllability behavior. This concise overview lays the groundwork for an in-depth investigation into the rules and factors governing the controllability of independent dynamic systems, illuminating their fundamental properties and consequences for scientific and engineering applications.

Let us consider a real-life example of a dynamical system and elucidate the concept of controllability. Consider an electrical circuit consisting of a resistor ($\mathcal{R}$) and a capacitor ($\mathcal{C}$) connected in series.

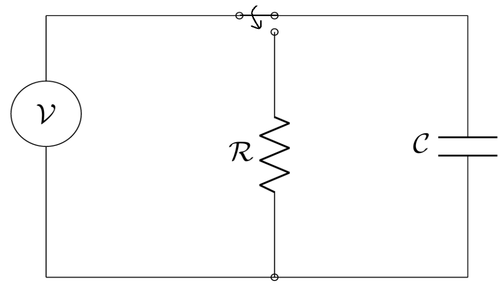

Figure 1.1: A simple $\mathcal{RC}$ circuit (resistor-capacitor circuit)

The capacitor is initially charged and then allowed to discharge through the resistor. By Ohm's law, the rate of decay is proportional to the voltage $\mathcal{V}$ and inversely proportional to the product of resistance $\mathcal{R}$ and capacitance $\mathcal{C}$, which together determine the decay time constant, the voltage across the capacitor ($\mathcal{V}(t)$) can be described using the following equation:

$$\frac{d\mathcal{V}(t)}{dt} = -\frac{\mathcal{V}(t)}{\mathcal{RC}} \tag{1.1}$$

Assume that at $t = 0$, the initial voltage across the capacitor is $\mathcal{V}(0) = \mathcal{V}_0$. Solving (1.1), with respect to the initial condition, we get

$$\mathcal{V}(t) = \mathcal{V}_0 e^{-\frac{t}{\mathcal{RC}}} \tag{1.2}$$

In particular, suppose that $\mathcal{R} = 1000\Omega$, $\mathcal{C} = 0.001F$ and $\mathcal{V}(0) = 10v$. Then $\mathcal{V}(t) = 10e^{-t}$.

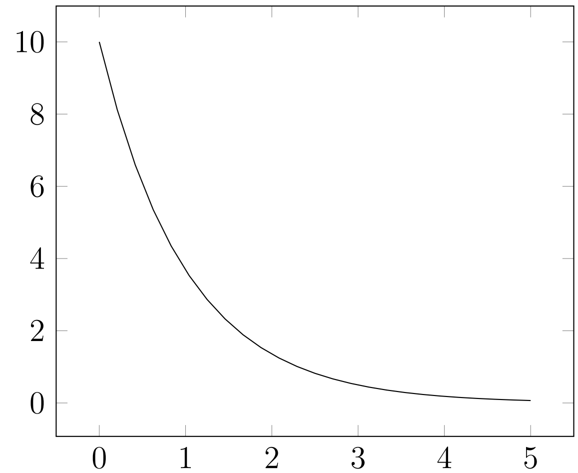

Figure 1.2: $\mathcal{V}(t) = 10e^{-t}$

Now, suppose that the $\mathcal{RC}$ circuit is supplied with an external current source, say $\mathcal{I}(t)$, as shown in the figure 1.3. Then by Kirchoff's law, (1.1) changes to

$$\frac{d\mathcal{V}(t)}{dt} = -\frac{\mathcal{V}(t)}{RC} + \frac{\mathcal{I}(t)}{\mathcal{C}} \tag{1.3}$$

Now, take $\mathcal{R} = 1000\Omega$, $\mathcal{C} = 0.001F$ and $\mathcal{V}(0) = 10v$ as earlier. Also, let $\mathcal{I}(t) = te^t$.

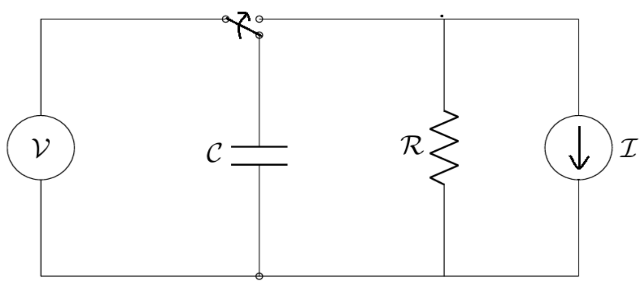

Figure 1.3: $\mathcal{RC}$ circuit with an external current source

Solving (1.3), we get

$$\mathcal{V}(t) = 11e^{-t} + t - 1 \tag{1.4}$$

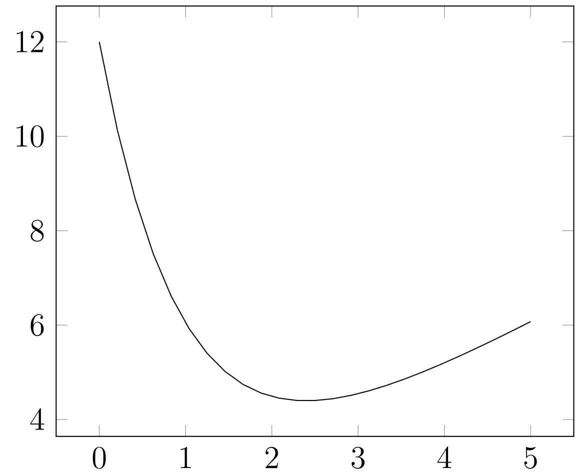

Figure 1.4: $\mathcal{V}(t) = 11e^{-t} + t - 1$

Observe how the voltage across the capacitor has changed as a result of the existence of an external current source. This suggests that we can alter the course of a dynamical system by adding a forcing factor. Using this concept, we can manipulate the dynamics of a system so that, at a particular time, its trajectory passes through a specific point.

Let us generalize the dynamical system in equation (1.1) to a broader scope and introduce what a control problem is. Consider a control system characterized by a differential equation of the form

$$\dot{x}(t) = ax(t) + bu(t), x(t_0) = x_0 \tag{1.5}$$

where a, b $(b \neq 0)$ are constants. The controllability problem is to check the existence of a forcing term or control function $u(t)$ such that the corresponding solution of the system will pass through a desired point $x(t_1) = x_1$.

Choose a differentiable function $z(t)$ satisfying $z(t_0) = x_0$ and $z(t_1) = x_1$. For example, take

$$z(t) = x_0 + \frac{(x_1 - x_0)}{t_1 - t_0}(t - t_0) \tag{1.6}$$

Clearly $z(t_0) = x_0$ and $z(t_1) = x_1$. Define a control term using the function z by

$$u = \frac{1}{b}\left[\dot{z} - az\right] \tag{1.7}$$

substituting in (1.5), we get

$$\dot{x} = ax + b\left\{\frac{1}{b}\left[\dot{z} - az\right]\right\}$$

This implies,

$$\dot{x} - \dot{z} = a(x - z)$$

That is,

$$\frac{d}{dt}(x - z) = a(x - z)$$

Also,

$$(x - z)(t_0) = x(t_0) - z(t_0) = x_0 - x_0 = 0$$

Taking $y(t) = x(t) - z(t)$, (1.5) becomes of the form,

$$\dot{y} = ay, \ y(0) = 0 \tag{1.8}$$

We know that the unique solution of (1.8) is $y(t) = x(t) - z(t) = 0$. That is, $x(t) = z(t)$ is the solution of the control system satisfying the required condition $x(t_0) = x_0$ and $x(t_1) = x_1$.

For example, consider the system given in (1.3), with the conditions $\mathcal{R} = 1000\Omega$, $\mathcal{C} = 0.001F$ and $\mathcal{V}(0) = 10v$. Now, suppose that we need the voltage across the capacitor to be $2v$ after 4 seconds. That is, $\mathcal{V}(4) = 2v$. In this case, $z(t)$ as defined in (1.6) can be found as follows;

$$\begin{aligned} z(t) &= x_0 + \frac{(x_1 - x_0)}{t_1 - t_0}(t - t_0) \\ &= 10 + \left(\frac{2 - 10}{4 - 0}\right)(t - 0) \\ &= 10 - 2t \end{aligned}$$

Now, define

$$\mathcal{I}(t) = 0.001(8 - 2t)$$

Then we get $\mathcal{V}(t) = 10 - 2t$ as a solution of (1.3). Thus we can say that (1.3) is an example for a controllable dynamical system. In fact, we have proven that any system of the form (1.5) is controllable with the control function defined as in (1.6). Note that, here the system is not only controllable but also trajectory controllable. That is, system can be steered along a given trajectory $z(t)$.

Now, consider an $n-$dimensional dynamical system defined on the time interval $[t_0, t_f]$ characterized by the following equation:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \ x(t_0) = x_0 \tag{1.9}$$

where, $x(t) \in \mathbb{R}^n$ is the state vector and $u(t) \in \mathbb{R}^m$ is the control input vector. $A(t) = [a_{ij}(t)] \in \mathbb{R}^{n\times n}$ and $B(t) = [b_{ij}(t)] \in \mathbb{R}^{n\times m}$ are continuous in some interval $[t_0, t_f]$ and are called state matrix and control matrix, respectively.

Remark 1.1. If the state and control matrices of (1.9) do not change with time, then the system is called linear time invariant(LTI) system. Otherwise, it is called linear time variant(LTV) system.

Let $\{x_0^i : i = 1, 2, \ldots, n\}$ be a basis of $\mathbb{R}^n$. For each i, let $\phi_i(t) \in \mathbb{R}^n$ be the unique solution to the homogeneous system

$$\dot{x}(t) = A(t)x(t) \tag{1.10}$$

with initial condition $x(t_0) = x_0^i$. Now, $\{\phi_i(t) : i = 1, 2, \ldots, n\}$ is a basis of the solution space of the homogeneous system (1.10). Consider the $n \times n$ matrix

$$\Phi(t) = [\phi_1(t) \mid \phi_2(t) \mid \ldots \mid \phi_n(t)] \tag{1.11}$$

with n linearly independent solutions of (1.10) as columns. $\Phi(t)$ is called *fundamental matrix solution*(Coddington and Levinson, 1955) and it satisfies $\dot{\Phi}(t) = A(t)\Phi(t)$. Clearly, $\Phi(t)$ is non-singular for each t. It is clear that, any matrix $\hat{\Phi}(t)$ is a fundamental matrix solution to the homogeneous system (1.10) if and only if $\hat{\Phi}(t)$ is a solution matrix to the correspondinding matrix differential equation $\dot{X}(t) = A(t)X(t)$ and the columns of $\hat{\Phi}(t)$ are linearly independent. For any non-singular matrix $M \in \mathbb{R}^{n\times n}$, consider the matrix $\Phi(t)M$. We have

$$\frac{d(\Phi(t)M)}{dt} = \dot{\Phi}(t)M = (A(t)\Phi(t))\, M = A(t)\,[\Phi(t)M]$$

Also, the columns of $\Phi(t)M$ are linearly independent. Thus, for any non-singular matrix $M \in \mathbb{R}^{n\times n}$, $\Phi(t)M$ is also a fundamental matrix solution of (1.10). Then, the *state transition matrix* of the homogeneous system is defined by

$$\Phi(t, t_0) = \Phi(t)\Phi^{-1}(t_0),\ t_0 \leq t \leq t_f < \infty \tag{1.12}$$

The state transition matrix $\Phi(t, t_0)$ has the following properties:

1) $\Phi(t, t) = I_n,\ \forall t \in [t_0, \infty)$, where I_n denote the $n \times n$ identity matrix.

2) $\Phi^{-1}(t, t_0) = \Phi(t_0, t)$

3) $\Phi(.,.)$ satisfies the semi-group property

$$\Phi(t,s) = \Phi(t,\tau)\Phi(\tau,s),\ \forall t_0 \le \tau \le s \le t_f < \infty$$

4) $\dot{\Phi}(t,t_0) = A(t)\Phi(t,t_0)$

5) $\Phi(t,t_0)$ is the unique solution of the matrix initial value problem

$$\dot{X}(t) = A(t)X(t),\ X(t_0) = I_n$$

Remark 1.2. The state transition matrix $\Phi(t,t_0)$ for (1.9) is given by the *Peano-Baker series*:

$$\begin{aligned}\Phi(t,t_0) = I_n &+ \int_{t_0}^{t} A(\sigma_1)d\sigma_1 + \int_{t_0}^{t} A(\sigma_1)\int_{t_0}^{\sigma_1} A(\sigma_2)d\sigma_2 d\sigma_1 \\ &+ \int_{t_0}^{t} A(\sigma_1)\int_{t_0}^{\sigma_1} A(\sigma_2)\int_{t_0}^{\sigma_2} A(\sigma_3)d\sigma_3 d\sigma_2 d\sigma_1 + \dots\end{aligned}$$

This series converges uniformly and absolutely for all $t_0 \le t \le t_f < \infty$ (See Chapter 1 in Brockett (2015)). If (1.9) is a LTI system, i.e., if $A(t) = A$, then the state transition matrix reduces to the matrix exponential given by

$$\Phi(t,t_0) = e^{A(t-t_0)} = I_n + A(t-t_0) + A^2\frac{(t-t_0)^2}{2!} + A^3\frac{(t-t_0)^3}{3!} + \dots$$

If $\Phi(t,t_0)$ is the state transition matrix of (1.10) with initial condition $x(t_0) = x_0$, then any future state $x(t)$ can be written by using the state transition matrix $\Phi(t,t_0)$ as

$$x(t) = \Phi(t,t_0)x_0$$

Hence the name transition matrix for $\Phi(t,t_0)$. Now, a solution to the non-homogeneous system (1.9) can be obtained by using the transition matrix as follows. Let $\Phi(t,t_0)$ be the transition matrix of the homogeneous system $\dot{x} = A(t)x$. Consider the transformation

$$z(t) = \Phi(t_0,t)x(t)$$

Then

$$x(t) = \Phi(t,t_0)z(t) \tag{1.13}$$

Differentiating with respect to t,

$$\dot{x}(t) = \dot{\Phi}(t, t_0)z(t) + \Phi(t, t_0)\dot{z}(t)$$

This implies that

$$\begin{aligned} A(t)x(t) + B(t)u(t) &= A(t)\Phi(t, t_0)z(t) + \Phi(t, t_0)\dot{z}(t) \\ &= A(t)x(t) + \Phi(t, t_0)\dot{z}(t) \end{aligned}$$

Thus, we have

$$B(t)u(t) = \Phi(t, t_0)\dot{z}(t)$$

and hence

$$\dot{z}(t) = \Phi(t_0, t)B(t)u(t)$$

Integrating over t_0 to t,

$$z(t) - z(t_0) = \int_{t_0}^{t} \Phi(t_0, \tau)B(\tau)u(\tau)d\tau$$

which implies,

$$z(t) = z(t_0) + \int_{t_0}^{t} \Phi(t_0, \tau)B(\tau)u(\tau)d\tau$$

Since $z(t_0) = x_0$,

$$z(t) = x_0 + \int_{t_0}^{t} \Phi(t_0, \tau)B(\tau)u(\tau)d\tau$$

Using (1.13), we have

$$x(t) = \Phi(t, t_0)x_0 + \Phi(t, t_0)\int_{t_0}^{t} \Phi(t_0, \tau)B(\tau)u(\tau)d\tau$$

By using the semi-group property, we have

$$x(t) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau)d\tau$$

as the required solution to the non-homogeneous system.

Definition 1.1 (Controllability)**.** The system (1.9) is controllable in a time interval $[t_0, t_f]$ if, given any two states $x_0, x_f \in \mathbb{R}^n$, there exists an admissible control function $u \in \mathcal{L}^2([t_0, t_f], \mathbb{R}^m)$, such that the corresponding solution of (1.9) with the initial condition

$x(t_0) = x_0$ also satisfies the desired final state $x(t_f) = x_f$.

From the definition, (1.9) is controllable if and only if there exists $u \in \mathcal{L}^2([t_0, t_f], \mathbb{R}^m)$ such that

$$x_f = x(t_f) = \Phi(t_f, t_0)x_0 + \int_{t_0}^{t_f} \Phi(t_f, \tau)B(\tau)u(\tau)d\tau$$

Then,

$$x_f - \Phi(t_f, t_0)x_0 = \int_{t_0}^{t_f} \Phi(t_f, \tau)B(\tau)u(\tau)d\tau$$

Denote $x_f - \Phi(t_f, t_0)x_0 = w$, then

$$w = \int_{t_0}^{t_f} \Phi(t_f, \tau)B(\tau)u(\tau)d\tau \tag{1.14}$$

Thus, the system (1.9) is controllable if and only if for every $w \in \mathbb{R}^n$, there exists $u \in \mathcal{L}^2([t_0, t_f], \mathbb{R}^m)$ such that (1.14) is satisfied. Define an operator $\mathcal{C} : \mathcal{L}^2([t_0, t_f]; \mathbb{R}^m) \to \mathbb{R}^n$ by

$$\mathcal{C}u = \int_{t_0}^{t_f} \Phi(t_f, \tau)B(\tau)u(\tau)d\tau \tag{1.15}$$

Thus, the system (1.9) is controllable if and only if the operator $\mathcal{C}$ is onto. Obviously, $\mathcal{C}$ is a bounded linear operator and $\mathcal{C}$ defines its adjoint operator $\mathcal{C}^* : \mathbb{R}^n \to \mathcal{L}^2([t_0, t_f], \mathbb{R}^m)$ in the following way:

$$\begin{aligned}
\langle \mathcal{C}^* v, u \rangle_{\mathcal{L}^2} &= \langle v, \mathcal{C}u \rangle_{\mathbb{R}^n}, \forall\, u \in \mathcal{L}^2([t_0, t_f], \mathbb{R}^m), v \in \mathbb{R}^n \\
&= \left\langle v, \int_{t_0}^{t_f} \Phi(t_f, \tau)B(\tau)u(\tau)d\tau \right\rangle_{\mathbb{R}^n} \\
&= \int_{t_0}^{t_f} \langle v, \Phi(t_f, \tau)B(\tau)u(\tau) \rangle_{\mathbb{R}^n} d\tau \\
&= \int_{t_0}^{t_f} \langle B^*(\tau)\Phi^*(t_f, \tau)v, u(\tau) \rangle_{\mathbb{R}^m} d\tau \\
&= \langle B^*(.)\Phi^*(t_f, .)v, u \rangle_{\mathcal{L}^2}
\end{aligned}$$

Hence, the adjoint operator of $\mathcal{C}$ is the linear operator $\mathcal{C}^* : \mathbb{R}^n \to \mathcal{L}^2([t_0, t_f]; \mathbb{R}^m)$, given by

$$(\mathcal{C}^* v)(t) = B^*(t)\Phi^*(t_f, t)v \tag{1.16}$$

The composition of $\mathcal{C}$ and $\mathcal{C}^*$ defines a bounded linear operator $\mathcal{C}\mathcal{C}^* : \mathbb{R}^n \to \mathbb{R}^n$ by,

$$\mathcal{C}\mathcal{C}^* v = \int_{t_0}^{t_f} \Phi(t_f, \tau) B(\tau) B^*(\tau) \Phi^*(t_f, \tau) v d\tau \tag{1.17}$$

Clearly, the operator $\mathcal{C}\mathcal{C}^*$ can be realized as a $n \times n$ matrix, called *Controllability Gramian* of the system (1.9) and is denoted by $\mathcal{W}(t_0, t_f)$. The following theorem relates controllability of (1.9) and the properties of linear operators $\mathcal{C}, \mathcal{C}^*$ and $\mathcal{C}\mathcal{C}^*$.

Theorem 1.1. *The following statements are equivalent:*

(i) The system (1.9) *is controllable.*

(ii) The operator $\mathcal{C}$ *is onto.*

(iii) The adjoint operator $\mathcal{C}^*$ *is one-one.*

(iv) The Controllability Grammian $W(t_0, t_f) = \mathcal{C}\mathcal{C}^*$ *is invertible.*

Proof. Clearly, $(i) \iff (ii)$ by definition of the operator $\mathcal{C}$ in (1.15).

Now, let us show $(ii) \Rightarrow (iii)$. Suppose that $\mathcal{C}$ is onto. We have to show that $\mathcal{C}^*$ is one-one. It is enough to show that $\mathcal{C}^* v = 0$ if and only if $v = 0$. Let $v \in \mathbb{R}^n$ such that $\mathcal{C}^* v = 0$. As $\mathcal{C}$ is onto, there exists $u \in \mathcal{L}^2([t_0, t_f] : \mathbb{R}^m)$ such that $\mathcal{C}u = v$. Then

$$\langle v, v \rangle = \langle \mathcal{C}u, v \rangle = \langle u, \mathcal{C}^* v \rangle = \langle u, 0 \rangle = 0$$

This implies that $v = 0$. Hence $\mathcal{C}^*$ is one-one.

To prove $(iii) \Rightarrow (iv)$, suppose that $\mathcal{C}^*$ is one-one. Let $v \in \mathbb{R}^n$ be such that $\mathcal{C}\mathcal{C}^* v = 0$. Then,

$$0 = \langle 0, v \rangle = \langle \mathcal{C}\mathcal{C}^* v, v \rangle = \langle \mathcal{C}^* v, \mathcal{C}^* v \rangle$$

This implies,

$$\|\mathcal{C}^* v\|_{\mathcal{L}^2}^2 = 0$$

and hence

$$\|\mathcal{C}^* v\|_{\mathcal{L}^2} = 0$$

We know that $\|\mathcal{C}^* v\|_{\mathcal{L}^2} = 0$ if and only if $\mathcal{C}^* v = 0$. Since $\mathcal{C}^*$ is one-one,

$$\mathcal{C}^* v = 0 \Rightarrow v = 0$$

Thus $\mathcal{C}\mathcal{C}^*$ is one-one. As $\mathcal{C}\mathcal{C}^*$ is a mapping from $\mathbb{R}^n$ to itself one-oneness implies that $\mathcal{C}\mathcal{C}^* = W(t_0, t_f)$ is invertible.

$(iv) \implies (i)$ Suppose that $\mathcal{C}\mathcal{C}^* = \mathcal{W}(t_0, t_f)$ is invertible.

Define a control function

$$u(t) = B^*(t)\Phi^*(t_f, t)\mathcal{W}^{-1}(t_0, t_f)[x_f - \Phi(t_f, t_0)x_0] \tag{1.18}$$

Using this control, the state of the system is given by

$$x(t) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, \tau)B(\tau)B^*(\tau)\Phi^*(t_f, \tau)\mathcal{W}^{-1}(t_0, t_f)[x_f - \Phi(t_f, t_0)x_0]d\tau$$

Then

$$x(t_0) = \Phi(t_0, t_0)x_0 + \int_{t_0}^{t_0} \Phi(t_0, \tau)B(\tau)B^*(\tau)\Phi^*(t_f, \tau)\mathcal{W}^{-1}(t_0, t_f)[x_f - \Phi(t_f, t_0)x_0]d\tau = x_0$$

and

$$\begin{aligned}
x(t_f) &= \Phi(t_f, t_0)x_0 + \int_{t_0}^{t_f} \Phi(t_f, \tau)B(\tau)B^*(\tau)\Phi^*(t_f, \tau)\mathcal{W}^{-1}(t_0, t_f)[x_f - \Phi(t_f, t_0)x_0]d\tau \\
&= \Phi(t_f, t_0)x_0 + \mathcal{W}(t_0, t_f)\mathcal{W}^{-1}(t_0, t_f)[x_f - \Phi(t_f, t_0)x_0] \\
&= \Phi(t_f, t_0)x_0 + x_f - \Phi(t_f, t_0)x_0 \\
&= x_f
\end{aligned}$$

Since, x_0 and x_f are arbitrary, the system is controllable. □

Remark 1.3. There may be multiple control functions that guide a system from its initial state x_0 to a desired final state x_f. Nonetheless, it can be easily shown that, among of all of those steering controller functions, the one described in (1.18) has the minimum $\mathcal{L}^2$ norm. Thus, (1.18) provides minimum energy control.

The conditions given in Theorem 1.1 for the controllability of the system (1.9) can be simplified for the time invariant case. That is, when $A(t) = A$ and $B(t) = B$ are not time dependent matrices. The condition was obtained in terms of the matrices A and B by the Hungarian-American electrical engineer, mathematician, and inventor *Rudolf E. Kálman(1930-2016)*. The condition is named after him as *Kalman's rank condition* for the controllability of LTI systems.

Theorem 1.2. *If the system* (1.9) *is LTI, then it is controllable if and only if the controllability matrix*

$$\mathcal{Q}(A,B) = [B|AB|\cdots|A^{n-1}B]$$

is of full rank. That is, $\mathcal{Q}(A,B) = n$*.*

Proof. Suppose that the system (1.9) is controllable. That is, $Rangespace(\mathcal{C}) = \mathbb{R}^n$. As $\mathcal{Q}(A,B)$ can be considered as a bounded linear operator from $\mathbb{R}^{mn} \to \mathbb{R}^n$, to show that $\mathcal{Q}$ is of full rank it is enough to prove that $Rangespace\,[\mathcal{Q}(A,B)] = \mathbb{R}^n$. Clearly, $Rangespace\,[\mathcal{Q}(A,B)] \subset \mathbb{R}^n$. Now, let $v \in \mathbb{R}^n$. By Theorem 1.1, there exists $u \in \mathcal{L}^2([t_0,t_f];\mathbb{R}^m)$ such that $\mathcal{C}u = v$

$$\begin{aligned} \mathcal{C}u = v &\implies \int_{t_0}^{t_f} \Phi(t_f,\tau)Bu(\tau)d\tau = v \\ &\implies \int_{t_0}^{t_f} e^{A(t_f-t)}Bu(\tau)d\tau = v \end{aligned}$$

Expanding $e^{A(t_f-t)}$ and by using Cayley-Hamilton theorem, we have

$$\int_{t_0}^{t_f} [P_0(\tau)I + P_1(\tau)A + ... + P_{n-1}(\tau)A^{n-1}]Bu(\tau)d\tau = v$$

where, each $P_i(\tau)$ is a polynomial function of τ that appears during the expansion of $e^{A(t_f-t)}$. This implies that $v \in Rangespace\,[\mathcal{Q}(A,B)]$. Therefore $\mathbb{R}^n \subset Rangespace\,[\mathcal{Q}(A,B)]$ and hence $Rank\,[\mathcal{Q}(A,B)] = n$.

Conversely suppose that the system (1.9) is not controllable. Then by Theorem 1.1 $\mathcal{W}(t_0,t_f)$ is not invertible and hence there exists $v \neq 0 \in \mathbb{R}^n$ such that $\mathcal{W}(t_0,t_f)v = 0$. This implies that $v^*\mathcal{W}(t_0,t_f)v = 0$. Therefore

$$\langle \mathcal{W}v, v\rangle = \left\langle \int_{t_0}^{t_f} e^{A(t_f-t)}BB^*u(\tau)e^{A^*(t_f-t)}v d\tau, v \right\rangle = 0$$

This implies that

$$\int_{t_0}^{t_f} v^*e^{A(t_f-t)}BB^*e^{A^*(t_f-t)}v = \int_{t_0}^{t_f} \|B^*e^{A^*(t_f-t)}v\|^2 = 0$$

As $B^*e^{A^*(t_f-t)}v$ is a continuous function on $[t_0,t_f]$, this implies

$$B^*e^{A^*(t_f-t)}v = 0, \forall\, t \in [t_0,t_f] \implies v^*e^{A(t_f-t)}B = 0, \forall\, t \in [t_0,t_f]$$

In particular, for $t = t_f$, $v^* B = 0$. Further, differentiating $v^* e^{A(t_f - t)} B$ w.r.t. t and evaluating at $t = t_f$, we get $v^* AB = 0$. Successively differentiating and evaluating at $t = t_f$, we get

$$v^* B = v^* AB = \cdots = v^* A^{n-1} B = 0$$

That is, $v \perp Rangespace([B|AB|\cdots|A^{n-1}B])$. This implies that $Rank[B|AB|\cdots|A^{n-1}B] < n$. Hence the result follows by contraposition. □

Let us consider an example to illustrate the result. Consider a spring mass damper system. Let m denote the mass, κ and α, respectively, denote the spring constant and the damping coefficient.

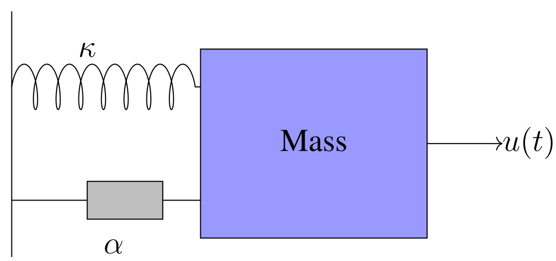

Figure 1.5: Spring Mass Damper system

Let $x(t)$ be the position of the mass at time t. Then $\dot{x}(t)$ gives the velocity and $\ddot{x}(t)$ is the acceleration of the mass at time t. The external force applied to the mass is denoted by $u(t)$.

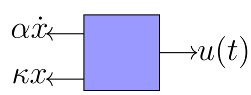

Figure 1.6: Forces acting on m

By Newton's second law of motion, the above system can be modeled as follows;

$$m\ddot{x} = u - \alpha\dot{x} - \kappa x \tag{1.19}$$

If we take $x_1 = x$ and $x_2 = \dot{x_1}$, the second order equation (1.19) can be reduced to a system of two first-order differential equations known as the state space representation as follows;

$$\dot{x_1} = x_2$$
$$\dot{x_2} = \frac{1}{m}(u - \alpha x_2 - \kappa x_1)$$

which further can be represented as

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{\kappa}{m} & -\frac{\alpha}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u$$

The controllability matrix $\mathcal{Q}$ is given by

$$\mathcal{Q}(A, B) = [B|AB] = \begin{bmatrix} 0 & \frac{1}{m} \\ \frac{1}{m} & \frac{-\alpha}{m^2} \end{bmatrix}$$

Clearly, $\mathcal{Q}(A, B)$ has rank 2 and hence system (1.19) is controllable. Controllability is an important notion in the analysis and design of LTI systems because it provides insight into the ability to steer the state of the system from any initial condition to any desired state within a certain time frame. The controllability requirements are critical for guaranteeing that all state variables may be modified using appropriate control inputs, allowing for the construction of effective control strategies to achieve desired system performance and stability. Later, Kalman proposed that an LTI system can be transformed into a specific canonical form, called controllability normal form, to facilitate the analysis of controllability properties. The controllability normal form is designed to have a block-triangular structure, making it easier to analyze and determine the controllability of the system. The transformation involves finding a similarity transformation matrix that diagonalizes the system's controllability matrix. The resulting controllability normal form provides valuable insights into the controllability properties of the system, allowing for a more straightforward assessment of its controllable modes(See Chapter 4 in (Terrell, 2009)). Using this notion, we have the following controllability results known as the *Popov-Belevitch-Hautus controllability test* named after the control scientists *V.M.Popov(1928-), V.Belevitch(1921-1999)* and *M.L.J.Hautus(1940-)*.

Theorem 1.3. *If the system* (1.9) *is LTI, then it is controllable if and only if for every* $\lambda \in \mathbb{C}$ *the only* $n \times 1$ *vector* v *that satisfies*

$$\begin{aligned} v^* A &= \lambda v^* \\ v^* B &= 0 \end{aligned} \tag{1.20}$$

is the zero vector, $v = 0$.

Proof. Suppose that there exists $v \neq 0$ such that (1.20) is satisfied. Then,

$$\begin{aligned} v^*\mathcal{Q}(A,B) &= v^*[B|AB|\cdots|A^{n-1}B] \\ &= [v^*B|v^*AB|\cdots|v^*A^{n-1}B] \\ &= [0|\lambda v^*B|\cdots|\lambda^{n-1}v^*B] = 0 \end{aligned}$$

which implies that $rank\,[\mathcal{Q}(A,B)] < n$. Hence, the system (1.9) is not controllable.

Conversely, suppose that system (1.9) is not controllable. That is, $rank\,[\mathcal{Q}(A,B)] = r < n$. By Kalman controllability decomposition(Terrell, 2009) there exists a non-singular matrix T such that

$$T^{-1}AT = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} = \hat{A} \text{ and } T^{-1}B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$$

Now, we will construct a non-zero v that satisfies (1.20). Let $\tilde{v}$ be an eigenvector of A_{22}^T corresponding to the eigenvalue λ. That is $A_{22}^T\tilde{v} = \lambda\tilde{v}$. This implies that $\tilde{v}^*A_{22} = \overline{\lambda}\tilde{v}^*$. As A_{22} is a real matrix both λ and $\overline{\lambda}$ are eigenvalues of A_{22} and because of the similarity of A and $\hat{A}$ both λ and $\overline{\lambda}$ are eigenvalues of A also. Now, define $v^* = \begin{bmatrix} 0_{1\times r} & \tilde{v}^* \end{bmatrix} T^{-1}$. Then,

$$\begin{aligned} v^*A &= v^*T \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} T^{-1} \\ &= \begin{bmatrix} 0_{1\times r} & \tilde{v}^* \end{bmatrix} T^{-1}T \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} T^{-1} \\ &= \begin{bmatrix} 0 & \tilde{v}^*A_{22} \end{bmatrix} T^{-1} \\ &= \begin{bmatrix} 0 & \lambda\tilde{v}^* \end{bmatrix} T^{-1} = \lambda v^* \end{aligned}$$

Also,

$$v^*B = \begin{bmatrix} 0_{1\times r} & \tilde{v}^*. \end{bmatrix} T^{-1}T \begin{bmatrix} B_1 \\ 0 \end{bmatrix} T^{-1} = 0$$

□

Observability Problem

Observability is another core notion in control systems theory that focuses on the ability to derive a system's internal state from its output measurements. A system is considered observable in control theory if its complete state can be uniquely inferred from the given output information. Because an observable system enables for reliable monitoring and assessment of its internal dynamics, it is critical in devising successful control strategies. Consider the system (1.9) with an output equation as follows;

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) \end{aligned} \tag{1.21}$$

where $y(t) \in \mathbb{R}^p$ and $C(t) \in \mathbb{R}^{p\times n}$ are the output vector and output matrix respectively. Now, we have the following formal definition for observability.

Definition 1.2 (Observability)**.** The system (1.21) is said to be observable over a time period $[t_0, t_f]$ if it is possible to determine uniquely the initial state $x(t_0) = x_0$ from the knowledge of the output $y(t)$ over the time period $[t_0, t_f]$.

Let $\Phi(t, t_0)$ be the state transition matrix of the homogeneous system $\dot{x}(t) = Ax(t)$. The unique solution is given by

$$x(t) = \Phi(t, t_0)x_0$$

Then, the observability problem can be written as

$$y(t) = C(t)x(t) = C(t)\Phi(t, t_0)x_0,\ t_0 \leq t \leq t_f$$

As we have seen in the case for controllability of (1.9), we define an operator $\mathcal{M} : \mathbb{R}^n \to \mathcal{L}^2\left([t_0, t_f] : \mathbb{R}^m\right)$ by,

$$(\mathcal{M}x_0)(t) = C(t)\Phi(t, t_0)x_0 \tag{1.22}$$

That is, $(\mathcal{M}x_0)(t) = y(t)$. The initial state is mapped to the observed function. As we need to uniquely determine x_0 from $y(.)$, the system (1.21) is observable if and only if $\mathcal{M}$ is one-one. Here, the adjoint operator of $\mathcal{M}$ is $\mathcal{M}^* : \mathcal{L}^2([t_0, t_f] : \mathbb{R}^m) \to \mathbb{R}^n$ given by

$$\mathcal{M}^* v = \int_{t_0}^{t_f} \Phi^*(\tau, t_0)C^*(\tau)v(\tau)d\tau \tag{1.23}$$

The observability Gramian $\mathcal{M}^*\mathcal{M} : \mathbb{R}^n \to \mathbb{R}^n$ is given by

$$\mathcal{M}^*\mathcal{M}v = \mathfrak{M}(t_0, t_f) = \int_{t_0}^{t_f} \Phi^*(\tau, t_0)C^*(\tau)C(\tau)\Phi(\tau, t_0)v dt \tag{1.24}$$

From any initial state x_0, we have a unique state given by

$$x(t) = \Phi(t, t_0)x_0$$

Thus, observability problem reduces to finding the unique initial state x_0 from the knowledge of y observed on $[t_0, t_f]$. Like Theorem 1.1, for the controllability of system (1.9) we have the following theorem for observability of system (1.20).

Theorem 1.4. *The following statements are equivalent:*

(i) The system (1.20) *is observable.*

(ii) The operator $\mathcal{M}$ is one-one.

(iii) The adjoint operator $\mathcal{M}^*$ is onto.

(iv) The Observability Gramian $\mathfrak{M}(t_0, t_f) = \mathcal{M}^*\mathcal{M}$ is invertible.

Proof. Proof is similar to that of Theorem 1.1. □

Some kind of interconnections between controllability and observability can be observed. This interconnection is called duality. To delve into the notion of duality we define the notion of adjoint systems.

Definition 1.3 (Adjoint Systems)**.** A system with state $x(t)$ is said to be adjoint to a system with state $p(t)$ if $\langle x(t), p(t)\rangle$ is a constant. That is, if $\frac{d}{dt}\langle x(t), p(t)\rangle = 0$.

Theorem 1.5. *The systems*

$$\dot{x}(t) = A(t)x(t) \tag{1.25}$$

and

$$\dot{p}(t) = -A^*(t)p(t) \tag{1.26}$$

are adjoint to each other.

Proof. By the product rule for differentiation concerning inner-product, we have

$$\begin{aligned}\frac{d}{dt}(\langle x(t), p(t)\rangle) &= \langle \dot{x}(t), p(t)\rangle + \langle x(t), \dot{p}(t)\rangle \\ &= \langle A(t)x(t), p(t)\rangle + \langle x(t), -A^*(t)p(t)\rangle \\ &= \langle x(t), A^*(t)p(t)\rangle + \langle x(t), -A^*(t)p(t)\rangle \\ &= \langle x(t), 0\rangle = 0\end{aligned}$$

Hence $\langle x(t), p(t)\rangle$ is a constant, proving that the systems (1.25) and (1.26) are adjoint to each other. □

The state transition matrices of the above systems are also related as shown in the following theorem.

Theorem 1.6. *If $\Phi(t, t_0)$ is the transition matrix of the system $\dot{x}(t) = A(t)x(t)$, then $\Phi^*(t_0, t)$ is the transition matrix of $\dot{p}(t) = -A^*(t)p(t)$.*

Proof. By using the properties of transition matrix, we have $I = \Phi(t, t_0)\Phi(t_0, t)$. Differentiating w.r.t. t,

$$\begin{aligned}0 &= \dot{\Phi}(t, t_0)\Phi(t_0, t) + \Phi(t, t_0)\dot{\Phi}(t_0, t) \\ &= A(t)\Phi(t, t_0)\Phi(t_0, t) + \Phi(t, t_0)\dot{\Phi}(t_0, t) \\ &= A(t) + \Phi(t, t_0)\dot{\Phi}(t_0, t)\end{aligned}$$

This implies that $\Phi(t, t_0)\dot{\Phi}(t_0, t) = -A(t)$ and hence $\dot{\Phi}(t_0, t) = -\Phi(t_0, t)A(t)$. Thus, we have

$$\frac{d[\Phi^*(t_0, t)]}{dt} = -A^*(t)\Phi^*(t_0, t)$$

Therefore, $\Phi^*(t_0, t)$ satisfies $\dot{p}(t) = -A^*(t)p(t)$. Further, $\Phi^*(t_0, t_0) = I$. Thus, $\Phi^*(t_0, t)$ is the transition matrix to the adjoint system $\dot{p}(t) = -A^*(t)p(t)$. □

Theorem 1.7. *Consider the linear control system*

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{1.27}$$

and the input-free observation system

$$\begin{aligned}\dot{x}(t) &= -A^*(t)x(t) \\ y(t) &= B^*(t)x(t)\end{aligned} \tag{1.28}$$

System (1.27) *is controllable if and only if adjoint system* (1.28) *is observable.*

Proof. Suppose that the adjoint system (1.28) is observable.

$$\begin{aligned}
\text{System (1.28) is observable} \iff & \mathfrak{M}(t_0, t_f) = \int_{t_0}^{t_f} [\Phi^*(t_0, \tau)]^* [B^*(\tau)]^* B^*(\tau)\Phi^*(t_0, \tau)d\tau \text{ is invertible} \\
\iff & \int_{t_0}^{t_f} \Phi(t_0, \tau)B(\tau)B^*(\tau)\Phi^*(t_0, \tau)d\tau \text{ is invertible} \\
\iff & \int_{t_0}^{t_f} \Phi(t_f, t_0)\Phi(t_0, \tau)B(\tau)B^*(\tau)\Phi^*(t_f, t_0)\Phi^*(t_0, \tau)d\tau \text{ is invertible}
\end{aligned}$$

as both $\Phi(t_f, t_0)$ and $\Phi^*(t_f, t_0)$ are invertible

$$\begin{aligned}
\iff & \int_{t_0}^{t_f} \Phi(t_f, \tau)B(\tau)B^*(\tau)\Phi^*(t_f, \tau)d\tau \text{ is invertible} \\
\iff & \mathcal{W}(t_0, t_f) \text{is invertible} \\
\iff & \text{System (1.27) is controllable}
\end{aligned}$$

Thus, system (1.27) is controllable if and only if adjoint system (1.28) is observable. □

The notion of duality asserts that if a linear system is controllable, it shares similar structural properties with its dual, observable system. This means that the matrices associated with controllability and observability exhibit analogous patterns. Understanding duality is essential in designing balanced and well-behaved control systems, ensuring that controllability and observability are appropriately matched for optimal performance and stability. The concept of duality aids in the translation of similar LTI system conditions from the case of controllability to the case of observability for adjoint systems.

Theorem 1.8. *If the system* (1.21) *is LTI, then it is observable if and only if the observability matrix*

$$\mathcal{O}(C, A) = rank \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

has full column rank. That is, $rank[\mathcal{O}(C, A)] = n$.

Proof. Suppose (C, A) is observable. Then, $(-A^*, C^*)$ is controllable and hence

$$rank[\mathcal{Q}(-A^*, C^*)] = n$$

That is,

$$rank\big[C^*| - A^*C^*|(A^*)^2C^*|\cdots|(-1)^{n-1}(A^*)^{n-1}C^*\big] = n$$

As a matrix and its transpose have the same rank, we get

$$rank\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n$$

Similarly, the converse follows. □

We can also obtain the following PBH observability condition for the LTI system (1.21) in terms of eigenvalues and eigenvectors.

Theorem 1.9. *If the system* (1.21) *is LTI, then it is observable if and only if for every complex* λ *the only* $1 \times n$ *vector* w *that satisfies*

$$\begin{aligned} Aw &= \lambda w \\ Cw &= 0 \end{aligned}$$

is the zero vector, $w = 0$*.*

Proof. Suppose that the system (1.21) is observable. Then, $(-A^*, C^*)$ is controllable. By Theorem 1.3, for any $\lambda \in \mathbb{C}$ the only solution to

$$\begin{aligned} w^*(-A^*) &= \lambda w^* \\ w^*(C^*) &= 0 \end{aligned}$$

is the zero vector. Taking conjugate transpose, for any $\lambda \in \mathbb{C}$ the only solution to

$$\begin{aligned} -Aw &= \lambda w \\ Cw &= 0 \end{aligned}$$

is $w = 0$. Similarly, the converse follows. □

Controllability refers to the ability to influence the behavior of a dynamical system by applying control inputs. Various notions of controllability, such as state controllability, structural controllability, and so on, are proposed in the literature, and controllability conditions for both linear and nonlinear systems are established by numerous authors. State controllability of a system deals with its ability to steer the state from an arbitrary initial state to a desired final state using suitable control functions, whereas, Lin's structural controllability(Lin, 1974) attempts to set some values to the nonzero parameters in the system matrices such that the resulting system is state controllable in the sense of Kalman. Controllability, whether state or structural, has been intensively investigated for a variety of systems, and numerous controllability criteria have been found during the last several decades(Callier and Nahum, 1975; Hautus, 1969; Lin, 1977; Linnemann, 1986; Rahmani and Mesbahi, 2007; Rahmani et al., 2009; Tanner, 2004; Tarokh, 1992). The majority of these discoveries pertain to single higher-dimensional control systems. When it comes to semi-linear and nonlinear dynamical control systems, particularly those with various impulses, delays in state and control variables the bibliography is not as extensive as that of linear systems(Joshi and George, 1989; Mirza and Womack, 1971, 1972; Sukavanam, 2000; Vidyasagar, 1972). Many researchers have focused their attention on such systems in recent decades, proposing various adequate conditions on system parameters, leading to conditions of controllability of semi-linear and non-linear systems.

1.2 Networked Systems

The need for networked systems has become paramount, owing to the increasing interconnectivity of our world. Networked system comprises of several components or subsystems that interact and collaborate to achieve common objectives. Because the behavior of one component can affect the entire network, this interconnectedness brings new challenges and opportunities for control theory. As a result, the mathematical underpinnings of control theory are critical in tackling the complexities of networked systems and assuring stability, performance, and reliability in the face of changing technological environments.

In general, representing complex systems necessitates the use of a group of separate systems linked together via an interconnection structure. The controllability problem of large-scale complex networked systems presents exciting research possibilities. These research include a variety of system elements such as structural complexity, node dynamics, interaction among nodes, and so on. The study of controllability of networked systems is gaining popularity because it has applications in many domains of science and technology(Bassett and Sporns, 2017; Farhangi, 2009; Gu et al., 2015; Müller and Schuppert, 2011; Wang and Chen, 2003; Wuchty, 2014). Depending upon the dynamics of the individual nodes, networked systems can be broadly divided into two; namely, homogeneous networks and heterogeneous networks. If all the individual nodes have the same dynamics, the networked system is said to be homogeneous and heterogeneous otherwise.

Let us consider an example to show the significance of the study of controllability of networked systems. We have seen the example of a $\mathcal{RC}$ circuit as a stand-alone control system in the previous section. Now, let us connect two such systems to obtain a networked system as follows:

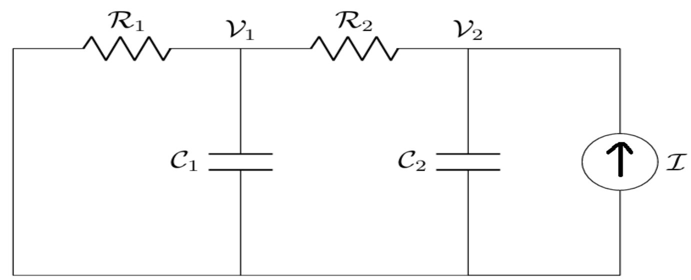

Figure 1.7: Network of Two $\mathcal{RC}$ Circuits

Then by Kirchoff's law, the voltage across the capacitors C_1 and C_2 are given by

$$\begin{aligned}\frac{d\mathcal{V}_1(t)}{dt} &= -\frac{\mathcal{V}_1}{\mathcal{C}_1\mathcal{R}_1} - \frac{\mathcal{V}_1}{\mathcal{C}_1\mathcal{R}_2} + \frac{\mathcal{V}_2}{\mathcal{C}_1\mathcal{R}_2} \\ \frac{d\mathcal{V}_2(t)}{dt} &= \frac{\mathcal{V}_1}{\mathcal{C}_2\mathcal{R}_2} - \frac{\mathcal{V}_2}{\mathcal{C}_2\mathcal{R}_2} + \frac{I}{\mathcal{C}_2}\end{aligned} \tag{1.29}$$

The system can be written in the form (1.9) of a stand-alone system as

$$\dot{\mathcal{V}}(t) = \begin{bmatrix}\dot{\mathcal{V}}_1(t) \\ \dot{\mathcal{V}}_2(t)\end{bmatrix} = \begin{bmatrix}-\left(\frac{1}{\mathcal{C}_1\mathcal{R}_1} + \frac{1}{\mathcal{C}_1\mathcal{R}_2}\right) & \frac{1}{\mathcal{C}_1\mathcal{R}_2} \\ \frac{1}{\mathcal{C}_2\mathcal{R}_2} & -\frac{1}{\mathcal{C}_2\mathcal{R}_2}\end{bmatrix}\begin{bmatrix}\mathcal{V}_1(t) \\ \mathcal{V}_2(t)\end{bmatrix} + \begin{bmatrix}0 \\ \frac{1}{\mathcal{C}_2}\end{bmatrix} I(t)$$

Then we can discuss the controllability of the given system (1.29) by using methods like Kalman's rank condition or PBH conditions. However, these results do not provide much information on individual systems or the connection between them. If we rewrite system (1.29) as

$$\begin{aligned}\dot{\mathcal{V}}_1(t) &= a_1\mathcal{V}_1 + h_1\mathcal{V}_1 + h_2\mathcal{V}_2 \\ \dot{\mathcal{V}}_2(t) &= a_2\mathcal{V}_2 + h_3\mathcal{V}_1 + b_1 I\end{aligned} \tag{1.30}$$

where $a_1 = -\frac{1}{\mathcal{C}_1\mathcal{R}_1}$, $a_2 = -\frac{1}{\mathcal{C}_2\mathcal{R}_2}$, $h_1 = -\frac{1}{\mathcal{C}_1\mathcal{R}_2}, h_2 = \frac{1}{\mathcal{C}_1\mathcal{R}_2}$, $h_3 = \frac{1}{\mathcal{C}_2\mathcal{R}_2}$ and $b_1 = \frac{1}{\mathcal{C}_2}$, we can identify the factors affecting the controllability of the system more easily. If we consider each $\mathcal{RC}$ circuit as an individual system we can observe that a_1, a_2 are the state matrices of the individual nodes, b_1 the control input matrix and h_1, h_2, h_3 represent the interconnections between the 'individual systems'. These are some of the factors that affect the controllability of a networked system. Therefore, if we consider system (1.29) as two individual systems connected together we can study the controllability of the given system in a detailed manner and we may be able to manipulate or control the system in a more feasible manner.

Now, we give a few numerical examples to demonstrate the complexities of studying networked systems. Consider a networked linear time invariant system with N nodes, where each node system is of dimension n. The dynamical system corresponding to the node i is described by

$$\begin{aligned} \dot{x}_i(t) &= Ax_i(t) + \sum_{j=1}^{N} \beta_{ij} H y_j(t) + d_i B u_i(t) \\ y_i(t) &= C x_i(t) \end{aligned} \tag{1.31}$$

where, for each $t \in [t_0, t_f]$, $x_i(t) \in \mathbb{R}^n$ is a state vector; $u_i(t) \in \mathbb{R}^m$ is an external control input vector; $y_i(t) \in \mathbb{R}^m$ is an output vector; $A \in \mathbb{R}^{n \times n}$ is the state matrix, $B \in \mathbb{R}^{n \times m}$ is the input matrix and $C \in \mathbb{R}^{m \times n}$ is the output matrix of node i. For a node i under control, $d_i = 1$, otherwise $d_i = 0$. $\beta_{ij} \in \mathbb{R}$ represents the communication strength between the nodes i and j. A communication from node j to node i ensures that $\beta_{ij} \neq 0$, otherwise $\beta_{ij} = 0$, for all $i, j = 1, 2, \ldots, N$. The inner coupling matrix describing the interconnections among the components $x_j, j = 1, 2, \ldots, N$ is denoted by $H \in \mathbb{R}^{n \times n}$. With output along with th state, system (1.31) becomes of the form

$$\dot{x}_i(t) = Ax_i(t) + \sum_{j=1}^{N} \beta_{ij} HC x_j(t) + d_i B u_i(t), \quad i = 1, 2, \ldots, N \tag{1.32}$$

Let $L = [\beta_{ij}] \in \mathbb{R}^{N \times N}$ represents the network topology and $D = diag\{d_1, d_2, \ldots, d_N\}$, the external input channels of the networked system (1.32). For example, if

$$L = \begin{bmatrix} 0 & 0 & \beta_{13} \\ \beta_{21} & 0 & 0 \\ \beta_{31} & \beta_{32} & 0 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{1.33}$$

then the network graph is given as in the following figure.

Also, let $X = \left[x_1^T, \ldots, x_N^T\right]^T$ denotes the network state and $U = \left[u_1^T, \ldots, u_N^T\right]^T$, the total

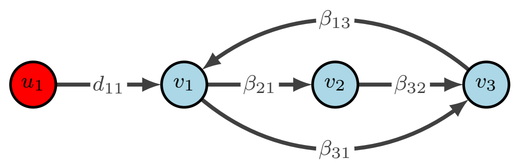

Figure 1.8: Network graph with L and D as given in (1.33).

external control input of the networked system. Then, using *Kronecker product*(Horn and Johnson, 1994) the networked system (1.32) can be rewritten in the compact form as

$$\dot{X}(t) = \Omega X(t) + \Psi U(t) \tag{1.34}$$

with

$$\Omega = I_N \otimes A + L \otimes HC, \ \Psi = D \otimes B$$

Example 1.1. Consider a homogeneous networked system with two individual nodes with dynamics;

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \ B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

The network topology matrix, inner-coupling matrix and the external control matrix are, respectively,

$$, \ L = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ H = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

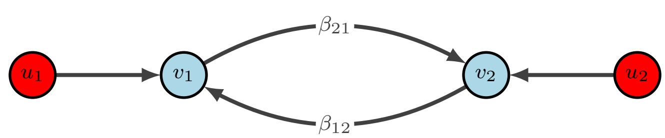

Figure 1.9: Networked system with two individual nodes and both nodes having external control inputs.

Observe that,

- ⊙ Both node 1 and 2 have external control inputs. That is, $d_1 = d_2 = 1$.
- ⊙ (A, B) is controllable as the controllability matrix

$$\mathcal{Q}(A, B) = \begin{bmatrix} B \mid AB \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

has rank 2.

- ⊙ (A, C) is observable as the observability matrix

$$\mathcal{O}(C, A) = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

has rank 2.

The system can be written in the compact form (1.34), where

$$\Omega = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \text{ and } \Psi = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

As the controllability matrix,

$$Q = [\Psi \mid \Omega\Psi \mid \Omega^2\Psi \mid \Omega^3\Psi] = \begin{bmatrix} 1 & 0 & 1 & 1 & 3 & 3 & 9 & 9 \\ 0 & 0 & 1 & 1 & 3 & 3 & 9 & 9 \\ 0 & 1 & 1 & 1 & 3 & 3 & 9 & 9 \\ 0 & 0 & 1 & 1 & 3 & 3 & 9 & 9 \end{bmatrix}$$

has rank 3, by Kalman's Rank Condition, we have that the networked system (Ω, Ψ) is not controllable even though the individual node system (A, B) is controllable.

Example 1.2. Consider a homogeneous networked system with two individual nodes with dynamics;

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

The network topology matrix, inner-coupling matrix and the external control matrix are, respectively,

$$L = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \; H = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Observe that,

- ⊙ Only node 1 has external control input. That is $d_1 = 1$ and $d_2 = 0$.

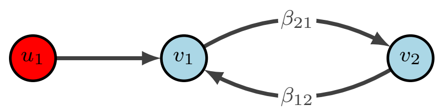

Figure 1.10: Networked system with two individual nodes and only node 1 having external control input.

- (A, B) is not controllable as the controllability matrix

$$\mathcal{Q} = \begin{bmatrix} B \mid AB \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$$

has rank 1.

The system can be written in the compact form (1.34), where

$$\Omega = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \text{ and } \Psi = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

As the controllability matrix,

$$Q = [\Psi \mid \Omega\Psi \mid \Omega^2\Psi \mid \Omega^3\Psi] = \begin{bmatrix} 1 & 2 & 4 & 8 \\ 1 & 2 & 4 & 9 \\ 0 & 0 & 1 & 4 \\ 0 & 1 & 3 & 8 \end{bmatrix}$$

has rank 4, by Kalman's Rank Condition, we have that the networked system (Ω, Ψ) is controllable even though the individual node system (A, B) is not controllable.

Example 1.3. Consider a homogeneous networked system with two individual nodes with dynamics;

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \ B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

The network topology matrix, inner-coupling matrix and the external control matrix are, respectively,

$$L = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ H = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

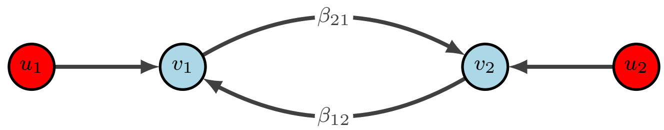

Figure 1.11: Networked system with two individual nodes and both nodes having external control inputs.

Observe that,

- ⊙ Both node 1 and 2 have external control inputs. That is, $d_1 = d_2 = 1$.
- ⊙ (A, C) is not observable as the observability matrix

$$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

  has rank 1.

The system can be written in the compact form (1.34), where

$$\Omega = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \text{ and } \Psi = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

As the controllability matrix,

$$Q = [\Psi \mid \Omega\Psi \mid \Omega^2\Psi \mid \Omega^3\Psi] = \begin{bmatrix} 1 & 0 & 2 & 1 & 4 & 3 & 8 & 7 \\ 1 & 0 & 1 & 1 & 2 & 2 & 4 & 4 \\ 0 & 1 & 1 & 2 & 3 & 4 & 7 & 8 \\ 0 & 1 & 1 & 1 & 2 & 2 & 4 & 4 \end{bmatrix}$$

has rank 4, by Kalman's Rank Condition, we have that the networked system (Ω, Ψ) is controllable even though the individual node system (A, C) is not observable.

We can see from the preceding examples that the controllability or observability of the individual system cannot ensure the controllability of the networked system. Individual node dynamics, network topology, and factors of such type play crucial roles in the controllability of a networked system. This makes studying controllability and observability of networked systems both difficult and interesting.

Research on the controllability of networked systems continues to be a dynamic and evolving field. Researchers are actively investigating how to manipulate and regulate the behavior of interconnected systems, including social networks, biological networks, and technological networks. The focus is on understanding the structural properties that influence the controllability of these systems, as well as developing practical algorithms and strategies for effective control. Ongoing efforts aim to address challenges related to scalability, adaptability, and robustness of control methods in the context of rapidly changing and complex network dynamics. The interdisciplinary nature of this research underlines its significance in tackling real-world problems arising in diverse domains, from power grids and transportation systems to information networks and beyond. Given the rapid technological advancements and the interconnectivity of systems, exploring the controllability in networked systems remains a vibrant and essential area of study.

1.3 Thesis outline and contributions overview

In this thesis, we investigate the controllability and observability of networked systems. The existing literature focuses mostly on results related to the controllability of homogeneous networked systems. The goal of this thesis is to provide a better understanding on the impact of individual dynamics, network topology and inner-coupling matrices on the controllability of heterogeneous networked systems and thereby providing verifiable controllability conditions for such systems. Another objective is to provide a method to make an uncontrollable system into a controllable system by manipulating its components, if possible. In **Chapter 2**, we give the preliminaries.

Chapter 3 introduces the concept of controllability for both homogeneous and heterogeneous networked systems. The controllability problem of homogeneous networked system (1.34) was first addressed by Wang et al. (Wang et al., 2016b). Wang et al. (Wang et al., 2016b) obtained a necessary and sufficient condition for controllability of such systems which involved solving matrix equations. Along with this result, Wang et al. (Wang et al., 2016b) derived some necessary conditions for the controllability of a homogeneous networked system which shows the effect of individual dynamics, inner coupling matrix and network topology on the controllability of the networked system. Among these, one particular result was that the observability of the individual node is necessary for the controllability of (1.34). Later Wang P. et al.(Wang et al., 2017b) and Xiang et al.(Xiang et al., 2019b) tried to provide conditions where the observability of the individual nodes is necessary for the controllability of heterogeneous networked systems. Xiang et al.(Xiang et al., 2019b) asserted that observability of each node is necessary for the controllability of a heterogeneous system when the state matrices are similar, the output matrices are scalar multiples of each other, and the rank of the input matrix of the networked system is less than the total number of nodes in the system. In this chapter, we provide an example to show that this result is not always true in general and also provide a corrected version of the result by Xiang et al.(Xiang et al., 2019b). Furthermore, we give some necessary conditions for the controllability of a heterogeneous networked system with aforementioned properties.(Ajayakumar and George, 2022b)

In **Chapter 4**, we discuss the controllability of heterogeneous networked systems with identical control input matrices. The controllability result obtained by Wang et al. (Wang et al., 2016b) does not give much information on the effect of node dynamics and network topology in the controllability of homogeneous networked system (1.34). Also, the result by Wang et al. involved solving matrix equations which makes the result a bit hard to verify. Later Hao et al.(Hao et al., 2018) studied the controllability of homogeneous networked systems with diagonalizable network topology matrix and obtained a set of conditions that are necessary and sufficient for the controllability of homogeneous networked systems. In this chapter, we extend this result for the controllability of heterogeneous networked systems with identical control input matrices and having triangularizable network topology matrix(Ajayakumar and George, 2023b). Here, we do not assume that the state matrices are identical. However, the controllability input matrix with the individual nodes is the same in all nodes. The results obtained give much more information regarding the involvement of node dynamics and network topology in the controllability of a networked system. Using this information we can manipulate the system to make an uncontrollable system into a controllable system. We will also show that the derived result will boil down to the result by Hao et al.(Hao et al., 2018), when the system is homogeneous and the network topology is triangularizable. In other words, we will show that the derived result is applicable to a larger class of systems when compared with the existing results in literature. Additionally, we derive controllability conditions for a few types of network topologies.

Chapter 5 deals with the controllability of heterogeneous networked systems with non-identical control input matrices and non-identical state matrices. In the previous chapter, the control input matrix in each node is considered to be the same in all nodes which is a limitation in many real-life applications. In this chapter, we tackle this limitation and extend the main result obtained in Chapter 4 to a larger class of systems. Also, a necessary and sufficient condition for the controllability of a networked system over traingular network topology is obtained. The obtained results are substantiated with numerical examples.

In **Chapter 6**, we discuss the controllability of networked systems in which each node have both linear and non-linear parts. The linear part of the networked system is assumed to be controllable and non-linear component in each node satisfies Lipschitz conditions. The controllability of the system is established by employing Generalized Banach Contraction Principle. Examples are provided to support the obtained results.

Chapter 7 deals with the notion of generic controllability of networked systems. In both homogeneous and heterogeneous networked system models discussed in previous chapters, we know apriori all the parameter values of the component matrices. However, in generic controllability, we do not know the weights of the interconnection links between the nodes. We only know the exact parameter values of system matrices for each node. Here, the system matrices are considered to be fixed for each node but the interconnection link between the nodes have unknown weights. Commault et al.(Commault and Kibangou, 2019) in 2019 proposed a set of conditions that are necessary and sufficient for the generic controllability of homogeneous networked systems. In this chapter, we investigate the generic controllability of heterogeneous networked systems and obtain some necessary conditions. Numerical examples are provided for the obtained results.

The thesis concludes in **Chapter 8** with the conclusions derived from our study and a proposal for future research.

Chapter 2

Preliminaries

In this chapter, we will give some basic definitions and results that are used in this thesis as prerequisites.

2.1 Tools from Matrix Analysis

2.1.1 Eigenvalues and Eigenvectors

Definition 2.1. Let $A \in \mathbb{C}^{n\times n}$. A non-zero vector $v \in \mathbb{C}^n$ is called a right eigenvector of A, if there exists a scalar λ such that $Av = \lambda v$. The scalar λ is called a right eigenvalue of A.

In other words, λ is a right eigenvalue of A, if there exists a non-zero vector v such that

$$(A - \lambda I)v = 0$$

This is only possible if $det(A - \lambda I) = 0$. Thus, we can say that λ is an eigenvalue of A if and only if it is a root of the polynomial equation $det(A - \lambda I) = 0$.

Definition 2.2. Let $A \in \mathbb{C}^{n\times n}$. A non-zero vector $v \in \mathbb{C}^n$ is called a left eigenvector of A, if there exists a scalar λ such that $v^T A = \lambda v^T$. The scalar λ is called a left eigenvalue of A.

Similar to right eigenvalue of A, λ is a left eigenvalue of A, if there exists a non-zero vector v such that

$$v^T(A - \lambda I) = 0$$

Taking transpose on both sides, we get

$$\left(v^T(A - \lambda I)\right)^T = (A - \lambda I)^T \left(v^T\right)^T = (A - \lambda I)^T v = 0$$

Again, this is possible only if $det(A - \lambda I)^T = 0$. As $det(A) = det(A^T)$, we can see that the *left and right eigenvalues of A are equal.*

Remark 2.1. Even-though the right and left eigenvalues of a matrix are the same the corresponding right and left eigenvectors need not be the same. For example, consider the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$$

We can see that the right eigenvectors of A are $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ corresponding to the eigenvalue 1 and $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ corresponding to the eigenvalue -1. However, the left eigenvectors of A are $v_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ corresponding to the eigenvalue 1 and $v_4 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ corresponding to the eigenvalue -1.

2.1.2 Similar Matrices

Definition 2.3. Two $n \times n$ matrices A_1 and A_2 are said to be similar if there exists a matrix T such that $T^{-1}A_1T = A_2$.

Example 2.1. Consider the matrices

$$A_1 = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} 10 & 15 \\ -5 & -8 \end{bmatrix}$$

For $T = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$, we have

$$T^{-1}A_1T = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 10 & 15 \\ -5 & -8 \end{bmatrix} = A_2$$

Thus, A_1 and A_2 are similar matrices.

Theorem 2.1. *(Horn and Johnson, 2012) Let A_1, A_2 be two $n \times n$ similar matrices, i.e., there exists an invertible matrix P such that $P^{-1}A_1P = A_2$. If (λ, v) is a right eigenpair of A_2, then (λ, Pv) is a right eigenpair of A_1.*

Proof. Since (λ, v) is a right eigenpair of A_2, we have $A_2 v = \lambda v$. Now,

$$P^{-1} A_1 P v = A_2 v = \lambda v \Rightarrow A_1 (Pv) = P(\lambda v) = \lambda (Pv).$$

That is, (λ, Pv) is a right eigenpair of A_1. □

Theorem 2.2. *(Horn and Johnson, 2012) Let A_1, A_2 be two $n \times n$ similar matrices, i.e., there exists an invertible matrix P such that $P^{-1} A_1 P = A_2$. If (λ, v) is a left eigenpair of A_2, then $(\lambda, v^T P^{-1})$ is a left eigenpair of A_1.*

Proof. Since (λ, v) is a right eigenpair of A_2, we have $v^T A_2 = \lambda v$. Now,

$$v^T P^{-1} A_1 P = v^T A_2 = \lambda v^T \Rightarrow \left(v^T P^{-1}\right) A_1 = (\lambda v^T) P^{-1} = \lambda (v^T P^{-1})$$

That is, $(\lambda, v^T P^{-1})$ is a left eigenpair of A_1. □

From Theorem 2.1 and 2.2, it is clear that similar matrices have the same eigenvalues. However, they need not have the same eigenvectors corresponding to an eigenvalue.

Definition 2.4. If $A \in \mathbb{C}^{n \times n}$ is similar to a diagonal matrix, then A is said to be diagonalizable. That is, A is diagonalizable if there exists a diagonal matrix D such that $P^{-1} A P = D$ for some invertible matrix $P \in \mathbb{C}^{n \times n}$.

Remark 2.2. The columns of P are formed by the eigenvectors of A, and the diagonal elements of D are the eigenvalues of A.

Example 2.2. Consider the matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$. If we take $P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, then we have

$$P^{-1} A P = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

Thus, A is diagonalizable.

Remark 2.3. Not all matrices are diagonalizable. For example, consider the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Suppose that A is diagonalizable. Then, there exist a matrix $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, with $ad - bc \neq 0$ such that

$$P^{-1} A P = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix} = D$$

This implies that,

$$\begin{bmatrix} ad+dc-bc & d^2 \\ -c^2 & -bc-dc+ad \end{bmatrix} = \begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix}$$

That is,

$$\begin{aligned} ad+dc-bc &= e \\ d^2 &= 0 \\ c^2 &= 0 \\ -bc-dc+ad &= f \end{aligned}$$

This gives $c = d = 0$, which is in contradiction with $ad - bc \neq 0$. Hence A is not diagonalizable.

Definition 2.5. If $A \in \mathbb{C}^{n\times n}$ is similar to a triangular matrix, then A is said to be triangularizable. That is, A is triangularizable if there exists a triangular matrix J such that $P^{-1}AP = J$ for some invertible matrix $P \in \mathbb{C}^{n\times n}$.

Definition 2.6. A Jordan block corresponding to λ of size m is an $m \times m$ matrix of the form $J_\lambda^m = \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda \end{pmatrix}$, where λ lies on the diagonal entries, 1 lies on the super diagonal and the remaining entries are all zeros.

Definition 2.7. A square matrix is said to be in Jordan canonical form, if it is a block diagonal matrix where each block is a Jordan block.

Remark 2.4. Every $n \times n$ complex matrix is similar to a matrix in Jordan canonical form. That is, every $n \times n$ complex matrix is triangularizable(See Chapter 4 in (George and Ajayakumar, 2024)).

Example 2.3. Consider the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. We have shown that the matrix is not diagonalizable in Remark 2.5. However, A is triangularizable as it is a Jordan block of order $2(J_1^2)$.

2.1.3 Kronecker Product of Matrices and its Properties

Definition 2.8. Let $A = \left[a_{ij}\right]_{m\times n}$ and $B = \left[b_{ij}\right]_{p\times q}$ be any two matrices, then the Kronecker product of A and B, denoted by $A \otimes B$, is the block matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{bmatrix}_{mp\times nq}$$

Example 2.4. Let $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix}$. Then the Kronecker product of A and B is given by

$$A \otimes B = \begin{bmatrix} 1 & 1 & 0 & 2 & 2 & 0 \\ -1 & 0 & 2 & -2 & 0 & 4 \\ 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -2 \end{bmatrix}$$

and the Kronecker product of B and A is given by

$$B \otimes A = \begin{bmatrix} 1 & 2 & 1 & 2 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 \\ -1 & -2 & 0 & 0 & 2 & 4 \\ 0 & 1 & 0 & 0 & 0 & -2 \end{bmatrix}$$

Clearly $A \otimes B \neq B \otimes A$.

The following properties of Kronecker product will be employed in this thesis.

Theorem 2.3. *(Horn and Johnson, 1994) Let* $A \in \mathbb{K}^{m\times n}, B \in \mathbb{K}^{p\times q}, C \in \mathbb{K}^{n\times k}$ *and* $D \in \mathbb{K}^{q\times r}$*. Then,*

$$(A \otimes B)(C \otimes D) = (AC \otimes BD)$$

Theorem 2.4. *(Horn and Johnson, 1994) If* $A \in \mathbb{K}^{m\times m}$ *and* $B \in \mathbb{K}^{n\times n}$ *are non-singular, then so is* $A \otimes B$ *and*

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$$

Theorem 2.5. *(Horn and Johnson, 1994) Let* A, B *and* C *be matrices be matrices of appropriate order. Then,* *(i)* $(A+B)\otimes C = A\otimes C + B\otimes C$

(ii) $A\otimes(B+C) = A\otimes B + A\otimes C$

(iii) $A\otimes B = 0$ *if and only if* $A = 0$ *or* $B = 0$.

2.2 Tools from Functional Analysis

Definition 2.9. Let V be a normed space and $\mathcal{K}$ be an operator such that $\mathcal{K} : V \to V$. Then $v \in V$ is a fixed point of $\mathcal{K}$ if $\mathcal{K}v = v$.

Definition 2.10. A mapping $\mathcal{K} : V \to V$ is said to be a contraction if there exists a real number $\alpha \in (0,1)$, such that

$$\| \mathcal{K}v - \mathcal{K}w \| \leq \alpha \| v - w \| \; \forall \, v, w \in V$$

Theorem 2.6 (Generalized Banach Contraction Principle)**.** *(Joshi and Bose, 1985) If V is a Banach space and $\mathcal{K} : V \to V$ is such that $\mathcal{K}^n : V \to V$ is a contraction for some n, then $\mathcal{K}$ has a unique fixed point. The unique fixed point can be computed iteratively by $v_{k+1} = \mathcal{K}^n v_k$, where v_0 is arbitrary.*

Definition 2.11. A proper algebraic variety is the zero set of some non-trivial polynomial with real coefficients in the n parameters of the system.

Definition 2.12. A property is said to be generic (or structural) if it is true for all values of the parameter vector, outside a proper algebraic variety in the parameter space $\mathbb{R}^n$.

Example 2.5. For example, consider a stand-alone system $\dot{x} = Ax + Bu$, where $A = [a_{ij}]_{n\times n}$ and $B = [b_{ij}]_{n\times m}$. We say that (A, B) is generically controllable if (A, B) is controllable for "almost all" values of $a_{ij}, i, j = 1, 2, \ldots, n$ and $b_{ij}, i = 1, 2, \ldots, n, \; j = 1, 2, \ldots, m$.

Consider a stand-alone system with state matrix $A = \begin{bmatrix} a_{11} & 0 \\ a_{21} & 0 \end{bmatrix}$ and control matrix $B = \begin{bmatrix} b_{11} \\ 0 \end{bmatrix}$. The system is generic controllable as the controllability matrix

$$\mathcal{Q} = [B \mid AB] = \begin{bmatrix} b_{11} & a_{11}b_{11} \\ 0 & a_{21}b_{11} \end{bmatrix}$$

has rank 2 for almost all values of a_{11}, a_{21} and b_{11} as the determinant of Q is $b_{11}^2 a_{21}$ and it is zero only when $b_{11} = 0$ or $a_{21} = 0$.

Consider a stand-alone system with state matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ 0 & 0 \end{bmatrix}$ and control matrix $B = \begin{bmatrix} b_{11} \\ 0 \end{bmatrix}$. The system is not generic controllable as the controllability matrix

$$\mathcal{Q} = [B \mid AB] = \begin{bmatrix} b_{11} & a_{11}b_{11} \\ 0 & 0 \end{bmatrix}$$

has rank 1 for any values of a_{11}, a_{12} and b_{11}.

2.3 Tools from Graph Theory

A graph G consists of a finite nonempty set V of objects called vertices or nodes and a set E of 2-element subsets of V called edges. The sets V and E are the vertex set and edge set of G, respectively. So a graph G is an ordered pair of two sets V and E represented as $G = (V, E)$. A graph can also be represented by a diagram in the plane as in the following figure where the vertices are represented by points or by small circles (open or solid) and whose edges are indicated by the presence of a line segment or curve between the two points in the plane corresponding to the appropriate vertices. If the edges in a graph have a direction associated with them, indicating a one-way relationship between vertices, then the graph is called a directed graph or digraph.

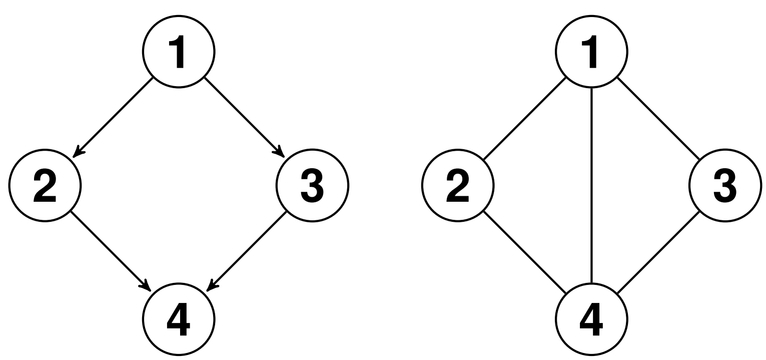

Figure 2.1: Example of a simple directed graph (left) and an undirected graph (right).

Definition 2.13. Let $G = (V, E)$ be a graph, where $V = \{v_1, v_2, \dots, v_k\}$ is the vertex set. A path in G from a vertex v_{i_0} to v_{i_q}, is a sequence of edges $(v_{i_0}, v_{i_1}), (v_{i_1}, v_{i_2}), \dots, (v_{i_{q-1}}, v_{i_q})$

such that $v_{i_l} \in V$ for $l = 0, 1, \dots, q$ and $(v_{i_{l-1}}, v_{i_l}) \in E$ for $l = 1, 2, \dots, q$. The vertices $v_{i_0}, v_{i_1}, \dots, v_{i_q}$ are said to be covered by the path.

Remark 2.5. When representing a networked system, if a path starts from a control node and ends at a state node, such paths are called control-state paths.

Definition 2.14. A stem is a control-state path which does not meet the same vertex twice.

Definition 2.15. A networked system is said to be control-connected if any state vertex is the end vertex of a stem.

Definition 2.16. A path $(v_{i_0}, v_{i_1}), (v_{i_1}, v_{i_2}), \dots, (v_{i_{q-1}}, v_{i_q})$ for which $v_{i_0} = v_{i_q}$ is called a circuit.

Definition 2.17. A cycle is a circuit which does not meet the same vertex twice, except for the initial/end vertex.

Remark 2.6. Two paths are mutually disjoint when they cover disjoint sets of vertices. When some stems and cycles are mutually disjoint, they constitute a disjoint set of stems and cycles.

Chapter 3

Controllability of Homogeneous and Heterogeneous Networked Systems

3.1 Introduction

In our increasingly interconnected world, understanding the controllability of networked systems is critical. Learning how to preserve the controllability of these complex systems becomes crucial as we shift from stand-alone systems to networked systems such as smart grids, transportation networks and social networks etc. Because these systems are interconnected, they present new difficulties and opportunities, necessitating a thorough understanding of how control signals and inter connections can be efficiently utilized to influence the behavior of the entire network. The topic of controllability in such a networked multi-agent system within the so-called 'leader-follower' framework was first addressed by Tanner (Tanner, 2004), where the problem is characterized as the classical state controllability of a single-input linear system. Tanner(Tanner, 2004) proposed several network topology requirements that assured the controllability of a set of nodes with a single leader by splitting the nodes into leaders and followers. Despite the fact that this criteria was derived for a single-leader system, they can be simply extended to multi-leader systems(see Rahmani et al. (Rahmani and Mesbahi, 2006)). Furthermore, Ji et al.(Ji et al., 2006) provides a necessary condition for multi-leader controllability based on the algebraic properties of a submatrix of the incidence matrix of the network. Hara et al.(Hara et al., 2009) investigated networks in which each node is a copy of the same single-input-single-output (SISO) system and discovered necessary and sufficient criteria for controllability and observability. The controllability criteria given by Tanner (Tanner, 2004) is not graph-theoretic in the sense that controllability cannot be derived directly based on network topology. Rahmani et al.(Rahmani and Mesbahi, 2006) examined the complex relationship between state controllability and graph symmetry, and then provided a suitable graph-theoretic criterion for determining uncontrollability. Many further studies followed based on the graph-theoretic properties of the network topology (Ji and Egerstedt, 2007; Lou and Hong, 2012; Martini et al., 2010; Mousavi and Haeri, 2016; Najafi and Shaikholeslam, 2013; Rahmani and Mesbahi, 2007; Rahmani et al., 2009).

It is evident that, in addition to network topology, the node system (nodal dynamics) is an important component influencing controllability. Wang et al.(Wang et al., 2016b) investigated networked MIMO LTI dynamical node systems with a directed and weighted topology without requiring an external control input on each subsystem. Some controllability constraints on network structure, node dynamics, external control inputs, and network topology are established so that effective criteria for defining the controllability of large-scale networked systems may be derived. Wang et al.(Wang et al., 2016b) derived a necessary and sufficient condition on the controllability of networked MIMO systems which involved finding the matrix solution of a set of matrix equations. It was proved that, under certain moderate conditions, node controllability and observability are necessary but not sufficient for networked system controllability. The importance of network topology for the controllability of the integrated networked system was also demonstrated. In particular, Wang et al.(Wang et al., 2016b) proved that when the rank of the input matrix of the networked system is less than the number of nodes of the networked system, the observability of each node is required for the controllability of a homogeneous networked system. Wang P. et al.(Wang et al., 2017b) further tried to extend the controllability results obtained by Wang et al.(Wang et al., 2016b) to heterogeneous networked systems and later, Xiang et al. (Xiang et al., 2019b) integrated these results along with a necessary and sufficient condition for the controllability of a heterogeneous networked system. When the state matrices are similar, the output matrices are scalar multiples of one another, and the rank of the input matrix of the networked system is less than the number of nodes in the system, Xiang et al. (Xiang et al., 2019b) proved that the observability of each node is required for the controllability of the heterogeneous system. However, this is not always the case. In Section 3.5, we provide a counter example to show that the result obtained by Xiang et al.(Xiang et al., 2019b) is not always true. We have also rectified this result by providing some additional conditions.

In Section 3.2, we formulate the homogeneous and heterogeneous networked system models under discussion. Controllability of homogeneous and heterogeneous networked systems are introduced and some available results in literature are discussed in Section 3.3 and Section 3.4 respectively with numerical examples to substantiate the results. In

Section 3.5, we derive some necessary conditions for the controllability of heterogeneous networked systems. Conclusions based on the study are given in Section 3.6.

3.2 Problem Formulation

Consider a linear time-invariant networked system with N nodes, where each node is an $n-$dimensional system. The dynamical system corresponding to the node i is described by

$$\begin{cases} \dot{x}_i(t) = Ax_i(t) + \sum_{j=1}^{N} \beta_{ij} H y_j(t) + d_i B u_i \\ y_i(t) = Cx_i(t) \end{cases} \tag{3.1}$$

where, $x_i(t) \in \mathbb{R}^n$ is the state vector of the i^{th} node; $u_i(t) \in \mathbb{R}^m$ is the external control input vector applied to the i^{th} node; $y_i(t) \in \mathbb{R}^m$ is the output vector of the i^{th} node; $A \in \mathbb{R}^{n\times n}$ is the state matrix, $B \in \mathbb{R}^{n\times m}$ is the input matrix and $C \in \mathbb{R}^{m\times n}$ is the output matrix of node i. If node i under external control, then $d_i = 1$, otherwise $d_i = 0$. $\beta_{ij} \in \mathbb{R}$ represents the communication strength between the nodes i and j. A communication from node j to node i ensures that $\beta_{ij} \neq 0$, otherwise $\beta_{ij} = 0$, for all $i, j = 1, 2, \dots, N$. The inner coupling matrix describing the interconnections among the components $x_j, j = 1, 2, \dots, N$ is denoted by $H \in \mathbb{R}^{n\times n}$. With control inputs, system (3.1) will take the form

$$\dot{x}_i(t) = Ax_i(t) + \sum_{j=1}^{N} \beta_{ij} HCx_j(t) + d_i B u_i(t), \quad i = 1, 2, \dots, N \tag{3.2}$$

Let $L = [\beta_{ij}] \in \mathbb{R}^{N\times N}$ represent the network topology and $D = diag\{d_1, d_2, \dots, d_N\}$, the external input channels of the networked system (3.2). Also, let $X = \left[x_1^T, \dots, x_N^T\right]^T$ denote the network state and $U = \left[u_1^T, \dots, u_N^T\right]^T$, the total external control of the networked system. Using Kronecker product, the homogeneous networked system (3.2) can be rewritten in the compact form as

$$\dot{X}(t) = \Omega X(t) + \Psi U(t) \tag{3.3}$$

with

$$\begin{aligned} \Omega &= I_N \otimes A + L \otimes HC \\ \Psi &= D \otimes B \end{aligned} \tag{3.4}$$

Now suppose that the individual nodes have different dynamics. That is, the networked system is heterogeneous. Let the dynamics of the i^{th} node is given by

$$\begin{cases} \dot{x}_i(t) = A_i x_i(t) + \sum_{j=1}^{N} \beta_{ij} H y_j(t) \\ y_i(t) = C_i x_i(t) \end{cases} \tag{3.5}$$

where A_i is the state matrix, B_i is the control input matrix and C_i is the output matrix of node i. All the other terms are as explained earlier. With control inputs, heterogeneous system (3.5) has the form

$$\dot{x}_i(t) = A_i x_i(t) + \sum_{j=1}^{N} \beta_{ij} H C_j x_j(t) + d_i B_i u_i(t), \quad i = 1, 2, \ldots, N \tag{3.6}$$

and the heterogeneous system can be written in the compact form as

$$\dot{X}(t) = \Omega X(t) + \Psi U(t) \tag{3.7}$$

where,

$$\begin{aligned} \Omega &= \Lambda + \Gamma \\ \Psi &= diag\{d_1 B_1, \ldots, d_N B_N\} \end{aligned} \tag{3.8}$$

where,

$$\Lambda = diag\{A_1, \ldots, A_N\}$$

and

$$\Gamma = \left[\beta_{ij} H C_j\right] \in \mathbb{R}^{nN \times nN}$$

3.3 Controllability of Homogeneous Networked Systems

In this section, we briefly discuss the controllability results obatained by Wang et al.(Wang et al., 2016b), for homogeneous networked systems. Over the past fifty years, a great deal of research has been done on the topic of system controllability. Thus far, numerous criteria have been established, such as distinct matrix rank conditions and graphical features(Davison and Wang, 1975; Davison, 1977; Gilbert, 1963; Glover and Silverman, 1976; Hautus, 1969; Kalman, 1960, 1962; Lin, 1974; Mayeda, 1981; Shields and Pearson, 1976). Notably, a large number of these controllability results are obtained with the supposition that each node is of state dimension one. Most real-world dynamical system networks, however, feature higher-dimensional node states, and many multi-input/multi-output (MIMO) nodes are linked together via multi-dimensional channels. Wang et al. (Wang et al., 2016b) studied the controllability of networked higher-dimensional systems with higher-dimensional connections for the MIMO setting and obtained the following necessary and sufficient condition.

Theorem 3.1. *(Wang et al., 2016b) The networked system* (3.2)-(3.4) *is controllable if and only if, for any complex number* s*, the matrix solution* $F \in \mathbb{C}^{N\times n}$ *of the simultaneous equations*

$$\begin{cases} D^T FB = 0 \\ L^T FHC = F(sI - A) \end{cases} \tag{3.9}$$

is $F = 0$*.*

Consider the following examples.

Example 3.1. (Ajayakumar and George, 2022a) Consider the homogeneous networked system with two individual nodes described by;

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, H = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

The network topology and the external control input channel matrices are given by

$$L = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Theorem 3.1 can be used to verify the controllability of the networked system. Let $F = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. Then,

$$\begin{aligned} D^T FB = 0 &\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ &\Rightarrow a_{11} + a_{12} = 0, a_{21} + a_{22} = 0 \\ &\Rightarrow a_{12} = -a_{11}, a_{22} = -a_{21} \end{aligned}$$

and for $s \in \mathbb{C}$,

$$L^T FHC = F(sI - A) \Rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & -a_{11} \\ a_{21} & -a_{21} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} a_{11} & -a_{11} \\ a_{21} & -a_{21} \end{bmatrix} \begin{bmatrix} s-1 & -1 \\ 0 & s-1 \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} a_{21} & 0 \\ a_{11} & 0 \end{bmatrix} = \begin{bmatrix} a_{11}(s-1) & -a_{11}s \\ a_{21}(s-1) & -a_{21}s \end{bmatrix}$$

When $s = 0$, $a_{11} = 1$ and $a_{21} = -1$, the matrix equation $L^T FHC = F(sI-A)$ is satisfied. Thus $F = 0$ is not a unique solution to the equations (3.9) and hence the networked system is not controllable.

We can also use Kalman's rank condition to see that the networked system is not controllable. The given system can be written in the compact form (3.3), where

$$\Phi = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } \Psi = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

The controllability matrix

$$Q(\Phi, \Psi) = [\Psi \mid \Phi\Psi \mid \Omega^2\Psi \mid \Omega^3\Psi] = \begin{bmatrix} 1 & 0 & 2 & 1 & 4 & 3 & 8 & 7 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 & 3 & 4 & 7 & 8 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

has rank 3 and hence the networked system is not controllable.

Example 3.2. (Ajayakumar and George, 2022a) Consider the homogeneous networked system with two individual nodes described by;

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, H = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

The network topology and the external control input channel matrices are given by

$$L = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Let $F = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. Then,

$$\begin{aligned} D^T F B = 0 \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \Rightarrow a_{12} = 0 & \end{aligned}$$

and for $s \in \mathbb{C}$,

$$\begin{aligned} L^T F H C = F(sI - A) \Rightarrow & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} s-1 & -1 \\ 0 & s-1 \end{bmatrix} \\ \Rightarrow & \begin{bmatrix} a_{22} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a_{11}(s-1) & -a_{11} \\ a_{21}(s-1) & -a_{21} + a_{22}(s-1) \end{bmatrix} \\ \Rightarrow & a_{11} = a_{21} = a_{22} = 0 \end{aligned}$$

Thus, $F = 0$ is the unique solution to the equations (3.9) and hence the networked system is controllable.

Here also we can use Kalman's rank condition to verify the controllability of the networked system. The given system can be written in the compact form (3.3), where

$$\Phi = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \text{ and } \Psi = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The controllability matrix

$$Q(\Phi, \Psi) = [\Psi \mid \Phi\Psi \mid \Omega^2\Psi \mid \Omega^3\Psi] = \begin{bmatrix} 0 & 0 & 1 & 0 & 2 & 0 & 3 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 3 & 0 \end{bmatrix}$$

has rank 4 and hence the networked system is controllable.

Although, the result is valid, it is not that easy to verify the controllability of a networked system using Theorem 3.1, as finding the matrix solution of Equation (3.9) is computationally demanding. Also, the result does not give much information regarding the effect of individual node dynamics, network topology, etc. in the controllability of the networked system. Along with Theorem 3.1, Wang et al.(Wang et al., 2016b) derived the following necessary conditions for controllability of the homogeneous networked system (3.2)-(3.4).

Theorem 3.2. *(Wang et al., 2016b) Suppose that the networked system* (3.2)-(3.4) *is controllable.*

(a) If there exists one node without incoming edges, it is necessary that (A, B) is controllable and moreover an external control input is applied onto this node which has no incoming edges.

(b) If there exists one node without external control inputs, it is necessary that (A, HC) is controllable.

(c) If the number of individual nodes is N and the number of nodes with external control is m with $N > m.rank(B)$, then it is necessary that (A, C) is observable.

(d) (L, D) is a controllable pair.

We can see that controllability and observability of the individual nodes are necessary for the controllability of networked systems under some moderate conditions, but they are not sufficient. Also, the pair of network topology matrix and the external input channel matrix must be a controllable pair for the controllability of the integrated networked system. Further, the controllability of networked system (3.2)-(3.4) was studied by Wang L et al.(Wang et al., 2017a) and obtained necessary and sufficient conditions for controllability where the higher dimensional states of the node systems are integrated into one dimensional input and and output. To state the result the following notations are needed. The set of nodes with external control inputs is denoted by $\mathfrak{U}$. That is,

$$\mathfrak{U} = \{i \mid d_i \neq 0,\ i = 1, 2, \ldots, N\} \tag{3.10}$$

Let $\sigma(A)$ denote the spectrum of the matrix A. For any $s \in \sigma(A)$, define the matrix set

$$\Gamma(s) = \Big\{ \Big[\alpha_1^T, \ldots, \alpha_N^T\Big] \mid \alpha_i^T \in \Gamma_1(s) \text{ for } i \notin \mathfrak{U},\ \alpha_i^T \in \Gamma_2(s) \text{ for } i \in \mathfrak{U} \Big\} \tag{3.11}$$

where

$$\Gamma_1(s) = \big\{ \xi \in \mathbb{C}^{1\times n} \mid \xi(sI - A) = 0 \big\} \tag{3.12}$$

and

$$\Gamma_2(s) = \left\{ \xi \in \mathbb{C}^{1\times n} \mid \xi B = 0, \xi \in \Gamma_1(s) \right\} \tag{3.13}$$

The following result gives us a glimpse how the network topology, individual node dynamics, inner interactions and external inputs influence the controllability of the whole networked system.

Theorem 3.3. *(Wang et al., 2017a) Suppose that* $\mid \mathfrak{U} \mid < N$*. Then the networked system* (3.2)-(3.4) *is controllable if and only if the following conditions hold.*

(a) (A, H) *is controllable;*

(b) (A, C) *is observable;*

(c) For any $s \in \sigma(A)$ *and* $\kappa \in \Gamma(s), \kappa L \neq 0 \; if \; \kappa \neq 0$*;*

(d) For any $s \notin \sigma(A)$*,* $rank[I - L\gamma \mid D\eta] = N$*, where* $\gamma = C(sI - A)^{-1}H$ *and* $\eta = C(sI - A)^{-1}B$*.*

The result is a not easy to verify. However, over certain network topologies, we have verifiable versions of the above theorem. Consider the following corollary.

Corollary 3.1. *(Wang et al., 2017a) Suppose that the network topology matrix* L *is a cycle. That is,* L *is of the form*

$$\begin{bmatrix} 0 & 0 & \dots & \beta_{1N} \\ \beta_{21} & 0 & \dots & 0 \\ \vdots & \ddots & & \vdots \\ 0 & \dots & \beta_{N(N-1)} & 0 \end{bmatrix}$$

Under the assumption that $B \in \mathbb{R}^{n\times 1}$ *,*$C \in \mathbb{R}^{1\times n}$*, and* $d_1 = 1, d_i = 0 \; \forall i = 2, 3, \dots, N$ *the networked system is controllable if and only if the following conditions hold.*

(a) (A, H) *is controllable;*

(b) (A, C) *is observable;*

(c) For any $s \notin \sigma(A)$*,* $rank(I - bHC(sI - A)^{-1}, B) = n$*, where* $b = \beta_{1N}\Pi_{i=1}^{N-1}\beta_{(i+1)i}\gamma^{N-1}$ *with* $\gamma = C(sI - A)^{-1}H$*.*

The following examples substantiate the efficiency of the above result.

Example 3.3. (Ajayakumar and George, 2022a) Consider a homogeneous networked system with 3 nodes, where the dynamics is given as follows;

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, H = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

The network topology and the external control input channel matrices are given by,

$$L = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

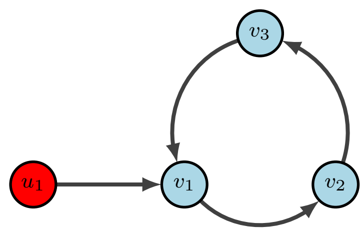

Figure 3.1: A cyclic network with 3 nodes and control input on node 1.

Clearly, $\sigma(A) = \{1, 1\}$. Now,

(a) (A, H) is controllable as the controllability matrix

$$\mathcal{Q}(A, H) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

has rank 2.

(b) (A, C) is observable as the controllability matrix

$$\mathcal{Q}(A, C) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

has rank 2.

(c) For any $s \neq 1$, we have $b = (s-1)^{-4}$ and

$$rank[I - bHC(sI - A)^{-1} \mid B] = rank\left(\begin{bmatrix} 1 & 0 & 1 \\ -(s-1)^{-5} & 1-(s-1)^{-5} & 0 \end{bmatrix}\right) = 2$$

Thus by Corollary 3.1, the given system is controllable.

Example 3.4. (Ajayakumar and George, 2022a) Consider a networked system with 3 identical nodes, where the dynamics of system is given as follows;

$$A = \begin{bmatrix} 1 & 8 & 7 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 4 & 3 & 6 \end{bmatrix}$$

The network topology matrix, inner-coupling matrix and the external control matrix are, respectively given by

$$L = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},\ H = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

For $s = 2 \notin \sigma(A) = \{2(3+\sqrt{7}), -2(\sqrt{7}-3), -3\}$,

$$rank[I - bHC(2I-A)^{-1} \mid B] = rank\left(\begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 \end{bmatrix}\right) = 2 < 3$$

Thus, by Corollary 3.1 the given system is not controllable.

All the above mentioned results are true for a homogeneous networked system. However, some additional conditions are required for the above results to hold true in the case of heterogeneous networked systems. We see this in the following section.

3.4 Controllability of Heterogeneous Networked Systems

In homogeneous networks, all components or nodes share similar attributes, whereas, in heterogeneous networks individual nodes need not posses same characteristics. Wang P. et al.(Wang et al., 2017b) obtained a necessary and sufficient condition on the controllability of the heterogeneous networked MIMO system (3.6)-(3.8).

Theorem 3.4. *(Wang et al., 2017b) The heterogeneous networked system* (3.6)-(3.8) *is controllable if and only if, for any complex numbers the solution* $\alpha_i \in \mathbb{C}^{1\times n}$ *of the simultaneous* *equations*

$$\begin{aligned} \alpha_i(sI_n - A_i) - \Sigma_{j=1,j\neq i}^{N} \beta_{ji}\alpha_j H C_i = 0 \\ d_i\alpha_i B_i = 0,\ i = 1, \ldots, N \end{aligned} \tag{3.14}$$

is $\alpha_i = 0,\ i = 1, 2, \ldots, N.$

Example 3.5. Consider a heterogeneous networked system with 2 nodes, where the state matrices, control matrices and output matrices are given by;

$$A_1 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix},\ A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix},\ B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ C_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, C_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

The network topology matrix, inner-coupling matrix and the external control matrix are, respectively given by

$$L = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Consider the simultaneous equations given in (3.14). Let $\alpha = \begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix}$ be a solution of

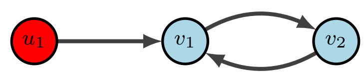

Figure 3.2: Networked system with two individual nodes and only node 2 having external control input.

(3.14), where $\alpha_i = \begin{bmatrix} \alpha_i^1 & \alpha_i^2 \end{bmatrix},\ i = 1, 2$. Then,

$$\begin{aligned} \begin{bmatrix} \alpha_1^1 & \alpha_1^2 \end{bmatrix} \begin{bmatrix} s-1 & -3 \\ 0 & s-1 \end{bmatrix} - \begin{bmatrix} \alpha_2^1 & \alpha_2^2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} &= \begin{bmatrix} 0 & 0 \end{bmatrix} \\ \begin{bmatrix} \alpha_2^1 & \alpha_2^2 \end{bmatrix} \begin{bmatrix} s-1 & -1 \\ 0 & s-3 \end{bmatrix} - \begin{bmatrix} \alpha_1^1 & \alpha_1^2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} &= \begin{bmatrix} 0 & 0 \end{bmatrix} \\ \begin{bmatrix} \alpha_1^1 & \alpha_1^2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= 0 \end{aligned}$$

Clearly, $\alpha_1^1 = 0$ and hence from above equations, we have

$$\begin{aligned} \begin{bmatrix} 0 & \alpha_1^2(s-1) \end{bmatrix} - \begin{bmatrix} \alpha_2^1 & \alpha_2^2 \end{bmatrix} &= \begin{bmatrix} 0 & 0 \end{bmatrix} \\ \begin{bmatrix} \alpha_2^1(s-1) & -\alpha_2^1 + \alpha_2^2(s-3) \end{bmatrix} - \begin{bmatrix} \alpha_1^2 & 0 \end{bmatrix} &= \begin{bmatrix} 0 & 0 \end{bmatrix} \end{aligned}$$

Solving, we get $\alpha_1^2 = \alpha_2^1 = \alpha_2^2 = 0$. Thus, both $\alpha_1 = \alpha_2 = 0$. Thus, by Theorem 3.4, the given system is controllable.

We can use Kalman's rank condition to verify the controllability of this system. The given system can be converted into the compact form (3.7)-(3.8), where,

$$\Phi = \begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 3 \end{bmatrix} \text{ and } \Psi = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The controllability matrix

$$Q(\Phi, \Psi) = [\Psi \mid \Phi\Psi \mid \Omega^2\Psi \mid \Omega^3\Psi] = \begin{bmatrix} 1 & 1 & 2 & 7 \\ 0 & 0 & 1 & 3 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

has rank 4 and hence the networked system is controllable.

Example 3.6. Consider a heterogeneous networked system with 2 nodes, where the state matrices, control matrices and output matrices are given by;

$$A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, C_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

The network topology matrix, inner-coupling matrix and the external control matrix are, respectively given by

$$L = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, H = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Consider the simultaneous equations given in (3.14). Let $\alpha = \begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix}$ be a solution of

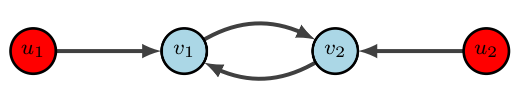

Figure 3.3: Networked system with two individual nodes and both nodes having external control input.

(3.14), where $\alpha_i = \begin{bmatrix} \alpha_i^1 & \alpha_i^2 \end{bmatrix}$, $i = 1, 2$. Then,

$$\begin{bmatrix} \alpha_1^1 & \alpha_1^2 \end{bmatrix} \begin{bmatrix} s-1 & -1 \\ 0 & s-1 \end{bmatrix} - \begin{bmatrix} \alpha_2^1 & \alpha_2^2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \alpha_2^1 & \alpha_2^2 \end{bmatrix} \begin{bmatrix} s & -1 \\ -1 & s-1 \end{bmatrix} - \begin{bmatrix} \alpha_1^1 & \alpha_1^2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \alpha_1^1 & \alpha_1^2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0$$

$$\begin{bmatrix} \alpha_2^1 & \alpha_2^2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$$

Clearly, $\alpha_1^1 = \alpha_2^2 = 0$ and hence from above equations, we have

$$\begin{bmatrix} 0 & \alpha_1^2(s-1) \end{bmatrix} - \begin{bmatrix} \alpha_2^1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \alpha_2^1 s & -\alpha_2^1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

Solving, we get $\alpha_2^1 = 0$. Thus, $\alpha_2 = 0$. However, when $s = 1$, α_1^2 can take any non-zero values. That is, there exist non-zero solutions of the form $\alpha = \begin{bmatrix} 0 & \alpha_1^2 & 0 & 0 \end{bmatrix}$, $\alpha_1^2 \in \mathbb{C}$ for the simultaneous equations (3.14) when $s = 1$. Hence, by Theorem 3.4, the given system is not controllable.

We can use Kalman's rank condition to verify the controllability of this system. The given system can be converted into the compact form (3.7)-(3.8), where,

$$\Phi = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \text{ and } \Psi = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

The controllability matrix

$$Q(\Phi, \Psi) = [\Psi \mid \Phi\Psi \mid \Omega^2\Psi \mid \Omega^3\Psi] = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 2 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 2 & 2 & 4 \\ 0 & 1 & 0 & 1 & 1 & 2 & 2 & 4 \end{bmatrix}$$

has rank 3 and hence the networked system is not controllable.

The preceding theorem is also computationally intensive and is quite similar to the PBH eigenvector test. Along with the above Theorem, Wang P. et al.(Wang et al., 2017b) extended the results in Theorem 3.2 of Wang et al.(Wang et al., 2016b) to the heterogeneous case and obtained the following results.

Theorem 3.5. *(Wang et al., 2017b) Suppose that the heterogeneous networked system* (3.6)-(3.8) *is controllable.*

(a) If there exists a node k without incoming edges, then it is necessary that (A_k, B_k) is controllable, and for any complex number $s \in \mathbb{C}$, the solution $\alpha_i \in \mathbb{C}^{1\times n}$ of both equations

$$\begin{cases} \alpha_i(sI_n - A_i) - \Sigma_{j=1, j\neq i, j\neq k}^{N} \beta_{ji}\alpha_j HC_i = 0 \\ d_i \alpha_i B_i = 0, \ i = 1, \ldots, N, i \neq k \end{cases}$$

is $\alpha_i = 0$.

(b) If there exists a node k without external control inputs, then it is necessary that $\begin{bmatrix} -\beta_{k1}HC_1 & -\beta_{k2}HC_2 & \ldots & sI - A_i & \ldots & -\beta_{kN}HC_N \end{bmatrix}$ has full rank.

(c) If the number of nodes with external control inputs is m, and $N > \Sigma_{i=1}^{m} rank(B_i)$, it is necessary that (A_i, C_i) is observable for $i = 1, 2, \ldots, N$.

(d) If $A_1 + s_0 HC_1 = \ldots = A_N + s_0 HC_N$ for all $s_0 \in \sigma(L)$, it is necessary that (L, D) is controllable.

However, Theorem 3.5(c) need not be true in general. Later, Xiang et al.(Xiang et al., 2019b) restated 3.5(c) as follows.

Theorem 3.6. *(Xiang et al., 2019b) Suppose $N > \Sigma_{i=1}^{\tilde{m}} rank(B_i)$($\tilde{m}$ is the number of external control inputs), $A_1, \ldots, A_N$ are similar to each other, and there exists $k_i \neq 0, i = 1, 2, \ldots, N$, such that $k_1 C_1 = k_2 C_2 = \ldots = k_N C_N$. For the heterogeneous networked system* (3.6)-(3.8) *to be controllable, it is necessary that (A_i, C_i) is observable for $i = 1, 2, \ldots, N$.*

In Section 3.5, we will give an example to show that the above result is also not true in general. Also, we provide a situation where the above result can be true. Along with the above result Xiang et al.(Xiang et al., 2019b) derived some necessary and sufficient conditions for some special class of heterogeneous networked systems. Consider a heterogeneous networked system (3.6)-(3.8) with

$$A_i = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_{i0} & -a_{i1} & -a_{i2} & \dots & -a_{i(n-1)} \end{bmatrix} \in \mathbb{R}^{n\times n},\ B_i = B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^n,\ C_i = C$$

and $d_i = 1, i = 1, 2, \dots, N$. Let $u_i = a_i^T x_i + d_{oi} u_{oi}$, where $a_i = \begin{bmatrix} a_{i0} & a_{i1} & a_{i2} & \dots & a_{i(n-1)}) \end{bmatrix}^T \in \mathbb{R}^n$, $u_{oi} \in \mathbb{R}$ is the external control input, and $d_{oi} = 1$ if i^{th} node is under control and is zero otherwise. Then (3.5) can be rewritten in the compact form as

$$\dot{x}_i(t) = Ax_i(t) + \sum_{j=1}^{N} \beta_{ij} HCx_j(t) + d_{oi} Bu_{oi}(t),\ i = 1, 2, \dots, N \tag{3.15}$$

where

$$A = A_i + Ba_i^T = A_i = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

If $X = \begin{bmatrix} x_1^T, \dots, x_N^T \end{bmatrix}^T$ denote the network state, $u_o = \begin{bmatrix} u_{o1}^T, \dots, u_{oN}^T \end{bmatrix}^T$, the total external control input of the networked system and $D = \{d_{oi}, \dots, d_{oN}\}$ the networked system can be written in the compact form.

$$\dot{X} = \Omega X + \Psi u_0 \tag{3.16}$$

where,

$$\Omega = I \otimes A + L \otimes HC$$

and

$$\Psi = D \otimes B$$

Let $C \in \mathbb{R}^n$. That is, input and output of each nodes are one dimensional. Let

$$\tilde{\nu} = \{i = 1, 2, \dots, \tilde{m} \mid d_{oi} = 1\},\ 1 \leq \tilde{m} \leq N$$

For $s \in \sigma\left(A_i + Ba_i^T\right)$, define the set

$$\Gamma(s) = \{(v_1, v_2, \ldots, v_N) \mid v_i \in \Gamma_1(s) \text{ for } v_i \notin \tilde{\nu}, \mid i \in \Gamma_2(s) \text{ for } i \in \tilde{\nu}\}$$

where,

$$\Gamma_1(s) = \left\{v \in \mathbb{C}^n \mid v^T\left(sI - A - Ba_i^T\right) = 0\right\},$$
$$\Gamma_2(s) = \left\{v \in \mathbb{C}^n \mid v^T B = 0,\ v \in \Gamma_1(s)\right\}$$

Xiang et al.(Xiang et al., 2019b) derived the following controllabily result for the particular class of networked systems defined in (3.15)-(3.16).

Theorem 3.7. *(Xiang et al., 2019b) Suppose that $|\tilde{\nu}| < N$. The networked system* (3.15)-(3.16) *with $C \in \mathbb{R}^n$ is controllable if and only if the following conditions hold:*

(i) $\left(A_i + Ba_i^T, H\right)$ *is controllable.*

(ii) $\left(A_i + Ba_i^T, C\right)$ *is observable.*

(iii) For $s \in \sigma\left(A_i + Ba_i^T\right)$ and $v \in \Gamma(s)$,

$$vL \neq 0 \text{ if } v \neq 0$$

and

(iv) For $s \notin \sigma\left(A_i + Ba_i^T\right)$,

$$rank[I - \gamma L \mid \eta D] = N$$

where, $\gamma = C\left(sI - A - Ba_i^T\right)^{-1} H$ and $\eta = C\left(sI - A - Ba_i^T\right)^{-1} B$.

We can clearly observe that there are no easily verifiable controllability results for a general heterogeneous networked system.

3.5 Necessary Conditions for Controllability of Heterogeneous Networked Systems

In this Section, we discuss some necessary conditions for the controllability of heterogeneous networked systems. First, we provide an example which shows that Theorem 3.6 by Xiang et al.(Xiang et al., 2019b) is not necessarily true in general.

Example 3.7. (Ajayakumar and George, 2022b) Consider a homogeneous networked system with 2 nodes, where the state matrices, control matrices and output matrices are given by;

$$A_1 = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix},\ A_2 = \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix},\ B_1 = B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ C_1 = C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

The network topology matrix, inner-coupling matrix and the external control matrix are, respectively,

$$L = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ H = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

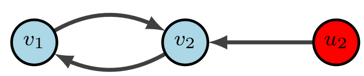

Figure 3.4: Networked system with two individual nodes and only node 2 having external control input.

- For $P = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$, we have $PA_1P^{-1} = A_2$. Thus, A_1 is similar to A_2.
- Here $C_1 = C_2$. That is, $k_1 = k_2 = 1$.
- The number of controlled nodes, $\tilde{m} = 1$, and hence

$$2 = N > \sum_{i=1}^{\tilde{m}} rank(B_i) = 1$$

Thus, all the conditions of Theorem 3.6 are satisfied. Now, the observability matrices

$$\mathcal{O}(A_1, C_1) = \begin{bmatrix} C_1 \\ C_1A_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

and

$$\mathcal{O}(A_2, C_2) = \begin{bmatrix} C_2 \\ C_2A_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$$

are of rank 1. Thus both (A_1, C_1) and (A_2, C_2) are not observable. The system can be written in the compact form (3.3), where

$$\Omega = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 3 & 1 & 0 \\ 1 & 0 & 3 & 0 \\ 1 & 0 & 2 & 1 \end{bmatrix}, \Psi = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Now, by Kalman's rank condition for controllability, the heterogeneous networked system is controllable as the controllability matrix,

$$\mathcal{Q}(\Omega, \Psi) = \begin{bmatrix} \Psi \mid \Omega\Psi \mid \Omega^2\Psi \mid \Omega^3\Psi \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 4 & 0 & 14 \\ 0 & 0 & 0 & 1 & 0 & 8 & 0 & 42 \\ 0 & 1 & 0 & 3 & 0 & 10 & 0 & 34 \\ 0 & 0 & 0 & 2 & 0 & 9 & 0 & 33 \end{bmatrix}$$

is of rank 4.

This discrepancy occured due to the fact that in the proof of Theorem 3.6, Xiang et al.(Xiang et al., 2019b) considered that similar matrices have identical eigenvectors for the same eigenvalue. However, this may not be the case always, which is evident from Theorem 2.1. We derived the following theorem incorporating this fact.

Theorem 3.8. *(Ajayakumar and George, 2022b) Suppose* $N > \sum_{i=1}^{\tilde{m}} rank(B_i)$. *Let* $A_1, A_2, \ldots, A_N$ *be similar to each other. That is, for each* A_i *there exists an invertible matrix* P_i^k *such that* $\left(P_i^k\right)^{-1} A_i P_i^k = A_k$*, for all* $i = 1, 2, \ldots, N$ *and* $k = 1, 2, \ldots, N$*. Also there exists* $k_i \neq 0, i = 1, 2, \ldots, N$*, such that* $k_1 C_1 = \cdots = k_N C_N$*. For the controllability of the heterogeneous networked system* (3.6) - (3.8)*, the observability of* (A_i, C_i) *is necessary for all* $i = 1, 2, \ldots, N$*, if the matrix* P_i^k *commutes with* C_i*.*

Proof. Assume that there exists a node i_0, such that (A_{i_0}, C_{i_0}) is unobservable. Then there exists $s \in \sigma(A_{i_0})$ and a non-zero vector $v_{i_0} \in \mathbb{C}^n$ such that

$$C_{i_0} v_{i_0} = 0$$

and

$$(sI_n - A_{i_0}) v_{i_0} = 0.$$

Let $\Omega_s = sI_{Nn} - \Omega = \left[\Omega_s^1 \mid \ldots \mid \Omega_s^N\right]$, where

$$\Omega_s^i = \left[-\beta_{1i}(HC_i)^T \mid \ldots \mid (sI_n - A_i)^T \mid \ldots \mid -\beta_{Ni}(HC_i)^T\right]^T.$$

Then $\Omega_s^{i_0} v_{i_0} = 0$, which implies that $rank\left(\Omega_s^{i_0}\right) \leq n-1$.
Since there exists matrices $P_k^{i_0}$ such that $\left(P_k^{i_0}\right)^{-1} A_k P_k^{i_0} = A_{i_0}$, for all $k = 1, 2, \ldots, N$, by Theorem 2.1, $P_k^{i_0} v_{i_0}$ is a right eigenvector of A_k with the eigenvalue s and as $P_k^{i_0}$ commutes with C_k for all $k = 1, 2, \ldots, N$, $C_k P_k^{i_0} v_{i_0} = 0$, where $k_1 C_1 = k_2 C_2 = \ldots = k_N C_N$ is employed. That is,

$$C_k P_k^{i_0} v_{i_0} = 0$$

and

$$(sI_n - A_k) P_k^{i_0} v_{i_0} = 0.$$

Therefore, $rank\left(\Omega_s^k\right) \leq n-1$ for all $k = 1, 2, \ldots, N$. That is, $rank\left(\Omega_s\right) \leq N(n-1)$. As $\sum_{i=1}^{\tilde{m}} rank(B_i) < N$, we have $rank(sI_{Nn} - \Omega, \Psi) < Nn$, which implies that the heterogeneous networked system (3.6) - (3.8) is not controllable. □

The following examples demonstrate the applicability of Theorem 3.8.

Example 3.8. (Ajayakumar and George, 2022b) Consider a homogeneous networked system with 2 nodes, where the state matrices, control matrices and output matrices are given by;

$$A_1 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \ A_2 = \begin{bmatrix} -1 & 0 \\ 4 & 3 \end{bmatrix}, \ B_1 = B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ C_1 = C_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

The network topology matrix, inner-coupling matrix and the external control input matrix are, respectively given by

$$L = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ H = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

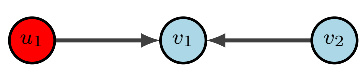

Figure 3.5: Networked system with two individual nodes and only node 1 having external control input.

⊙ For $P_1^2 = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$, we have $PA_1P^{-1} = A_2$. Thus, A_1 and A_2 are similar.

⊙ Clearly, $C_1 = C_2$. That is, $k_1 = k_2 = 1$.

⊙ The number of controlled nodes, $\tilde{m} = 1$, and hence

$$2 = N > \sum_{i=1}^{\tilde{m}} rank(B_i) = 1$$

⊙ Also, $P_1^2 = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$ and $P_2^1 = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$ commutes with C_1, C_2.

Thus, all the conditions of the Theorem 3.8 are satisfied. Now,

$$\mathcal{O}(A_1, C_1) = \begin{bmatrix} C_1 \\ C_1 A_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 3 & 3 \\ 0 & 0 \end{bmatrix}$$

is of rank 1. Hence (A_1, C_1) is not observable. Then by Theorem 3.8 the system is not controllable. We can verify this using Kalman's rank condition for controllability. The system can be written in the compact form as in Equation (3.7), where

$$\Omega = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 4 & 3 \end{bmatrix} \text{ and } \Psi = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Then the system is not controllable as the controllability matrix

$$\mathcal{Q}(\Omega, \Psi) = \begin{bmatrix} \Psi \mid \Omega\Psi \mid \Omega^2\Psi \mid \Omega^3\Psi \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 & 5 & 0 & 13 & 0 \\ 0 & 0 & 2 & 0 & 4 & 0 & 14 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is of rank 2.

Observe that the eigenvectors of A_1 and A_2 in Example 3.8, corresponding to the eigenvalue 3 are distinct. The eigenvectors of A_1 and A_2 corresponding to the eigenvalue 3 are $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, respectively. Theorem 3 of Xiang et al.(Xiang et al., 2019b) is true, if the associated eigenvectors of the same eigenvalues are identical for the similar state matrices $A_1, \ldots, A_N$. It is clear that, this result cannot be used to show the uncontrollability of the system in Example 3.8 as the identical eigenvector for same eigenvalue criteria is not satisfied.

Example 3.9. (Ajayakumar and George, 2022b) Consider a homogeneous networked system with 2 nodes, where the state matrices, control matrices and output matrices are given by;

$$A_1 = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}, \; A_2 = \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}, \; B_1 = B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; C_1 = C_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The network topology matrix, inner-coupling matrix and the external control input matrix are, respectively given by

$$L = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

- ⊙ A_1 and A_2 are the same matrices from Example 3.8 and we have seen that they are similar.
- ⊙ Clearly, $C_1 = C_2$. That is, $k_1 = k_2 = 1$.
- ⊙ The number of controlled nodes, $\tilde{m} = 1$, and hence

$$2 = N > \sum_{i=1}^{\tilde{m}} rank(B_i) = 1$$

- ⊙ Also, $P_1^2 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ and $P_2^1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ commutes with C_1, C_2.

Thus, all the conditions of Theorem 3.8 are satisfied. The system can be written in the compact form as in (3.7), where,

$$\Omega = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 2 & 3 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix} \text{ and } \Psi = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Then the networked system is controllable as the controllability matrix,

$$\mathcal{Q}(\Omega, \Psi) = \begin{bmatrix} \Psi \mid \Omega\Psi \mid \Omega^2\Psi \mid \Omega^3\Psi \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 2 & 0 & 10 \\ 0 & 0 & 0 & 1 & 0 & 6 & 0 & 31 \\ 0 & 1 & 0 & 3 & 0 & 8 & 0 & 27 \\ 0 & 0 & 0 & 2 & 0 & 9 & 0 & 26 \end{bmatrix}$$

is of rank 4. Clearly both (A_1, C_1) and (A_2, C_2) are observable.

If $D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, with all other matrices unchanged, the matrix can be written in the compact form as in Equation (3.7), where

$$\Omega = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 2 & 3 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix} \text{ and } \Psi = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Then, the system is not controllable as the matrix

$$\mathcal{Q}(\Omega, \Psi) = \begin{bmatrix} \Psi \mid \Omega\Psi \mid \Omega^2\Psi \mid \Omega^3\Psi \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 8 & 0 & 26 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is of rank 2. Here, both (A_1, C_1) and (A_2, C_2) are also observable.

The above examples show that the observability of (A_i, C_i) for all $i = 1, 2, \ldots, N$ is a necessary condition, but not sufficient. We can waive of the requirement that the state matrices $A_1, \ldots, A_N$ are similar and strengthen Theorem 3.4 of Xiang et al.(Xiang et al., 2019b), as follows also.

Theorem 3.9. *(Ajayakumar and George, 2022b) Suppose* $N > \sum_{i=1}^{\tilde{m}} rank(B_i)$ *and there exists* $k_i \neq 0, i = 1, 2, \ldots, N$, *such that* $k_1 C_1 = \cdots = k_N C_N$. *If the state matrices* $A_1, \ldots, A_N$ *have a common eigenpair* (s_0, v) *with* $C_i v = 0$ *for some* $i \in \{1, 2, \ldots, N\}$, *then the heterogeneous networked system* (3.6)-(3.8) *is uncontrollable.*

Proof. Suppose that the state matrices $A_1, \ldots, A_N$ have a common eigenpair (s_0, v) with $C_i v = 0$ for some $i \in \{1, 2, \ldots, N\}$. As there exist $k_i \neq 0, i = 1, 2, \ldots, N$, such that $k_1 C_1 = \cdots = k_N C_N$, we get $C_i v = 0$ for all $i = 1, 2, \ldots, N$. Also $(s_0 I_n - A_i)v = 0$ for all $i = 1, 2, \ldots, N$. Therefore, if we consider $\Omega_s = sI_{Nn} - \Omega = \left[\Omega_s^1 \mid \ldots \mid \Omega_s^N\right]$, where,

$$\Omega_s^i = \left[-\beta_{1i}(HC_i)^T \mid \ldots \mid (sI_n - A_i)^T \mid \ldots \mid -\beta_{Ni}(HC_i)^T\right]^T.$$

Then $\Omega_s^i v = 0$ for all $i = 1, 2, \ldots, N$. This implies that $rank\,(\Omega_s^i) \leq n - 1$ for all $i = 1, 2, \ldots, N$ and hence $rank\,(\Omega_s) \leq N(n-1)$. As $\sum_{i=1}^{\tilde{m}} rank(B_i) < N$, we have $rank(sI_{Nn} - \Omega, \Psi) < Nn$, which implies that the heterogeneous networked system (3.6) - (3.8) is not controllable. □

Example 3.10. (Ajayakumar and George, 2022b) Consider a homogeneous networked system with 2 nodes, where the state matrices, control matrices and output matrices are given by;

$$A_1 = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix},\ A_2 = \begin{bmatrix} 1 & 5 \\ 0 & 4 \end{bmatrix},\ B_1 = B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ C_1 = C_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

The network topology matrix, inner-coupling matrix and the external control input matrix are, respectively given by

$$L = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ H = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

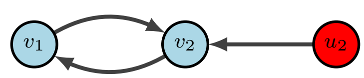

Figure 3.6: Networked system with two individual nodes and only node 2 having external control input.

- ⊙ Here $\tilde{m} = 1$ and

$$2 = N > \sum_{i=1}^{\tilde{m}} rank(B_i) = 1$$

- ⊙ Clearly, $C_1 = C_2$. That is, $k_1 = k_2$.

Thus, all the requirements of Theorem 3.9 are satisfied. Observe that, both A_1 and A_2 have 1 as an eigenvalue with $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as an eigenvector. Also, $C_1 v = C_2 v = 0$. Then by Theorem 3.9, the system is not controllable. We can verify this using Kalman's rank condition for controllability. The system can be written in the compact form as in Equation

(3.7), where,

$$\Omega = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 3 & 0 & 1 \\ 0 & 1 & 1 & 5 \\ 0 & 1 & 0 & 4 \end{bmatrix} \text{ and } \Psi = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Then the system is not controllable as the controllability matrix,

$$\mathcal{Q}(\Omega, \Psi) = \begin{bmatrix} \Psi \mid \Omega\Psi \mid \Omega^2\Psi \mid \Omega^3\Psi \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is of rank 1.

We can obtain a corrected version of Theorem 3.6 of Xiang et al.(Xiang et al., 2019b) as a corollary of Theorem 3.9.

Corollary 3.2. *(Ajayakumar and George, 2022b) Suppose $N > \sum_{i=1}^{\tilde{m}} rank(B_i)$, and the matrices $A_1, \ldots, A_N$ be similar to each other, where, the associated eigenvectors of the same eigenvalue are identical and there exist $k_i \neq 0, i = 1, 2, \ldots, N$, such that $k_1 C_1 = \cdots = k_N C_N$. For the heterogeneous networked system (3) to be controllable, it is necessary that (A_i, C_i) is observable for all $i = 1, 2, \ldots, N$.*

Proof. Suppose that (A_i, C_i) is not observable for some node i, say i_0. Then there exists a complex number $s_0 \in \sigma(A_{i_0})$ and a non-zero vector $v \in \mathbb{C}^n$ such that

$$C_{i_0} v = 0$$

and

$$(s_0 I_n - A_{i_0}) v = 0.$$

As the eigenvectors of the same eigenvalue are identical for the state matrices , (s_0, v) is a common eigenpair for all $A_1, \ldots, A_N$. Then by Theorem theorem3.8, the heterogeneous networked system (3) is uncontrollable. □

3.6 Conclusions

This chapter presents the concept of networked system controllability and explored some of the existing controllability results in the literature for both homogeneous and heterogeneous systems. Wang et al.(Wang et al., 2016b) examined networked MIMO LTI dynamical node systems with a directed and weighted topology that did not require an external control input for each subsystem. Some controllability constraints are set on network topology, node dynamics, external control inputs, and inner coupling matrices, allowing effective criteria for checking the controllability of large-scale networked systems. Wang et al.(Wang et al., 2016b) developed a necessary and sufficient condition on the controllability of networked MIMO systems by solving a series of equations. It was demonstrated that node controllability and observability are necessary but not sufficient for networked system controllability under certain moderate conditions. Later, Wang P. et al.(Wang et al., 2017b) and Xiang et al.(Xiang et al., 2019b) studied the controllability of heterogeneous networked systems and obtained some controllability results.

Xiang et al.(Xiang et al., 2019b) derived a necessary condition for the controllability of a special type of heterogeneous networked system, which states that the observability of each node is necessary for the controllability of a heterogeneous networked system satisfying: 1) similar state matrices; 2) that output matrices are scalar multiples of one another; and 3) that the rank of the input matrix of the networked system is less than the number of nodes in the system. In this chapter, we have given a counterexample to show that this claim is not true in general. We were able to identify the discrepancy and rectify it in the form of Theorem 3.8 and 3.9. Furthermore, we have derived some necessary conditions for the controllability of a heterogeneous networked system with aforementioned properties. The obtained results are substantiated with numerical examples.

Chapter 4

Controllability of Heterogeneous Networked Systems with Identical Control Input Matrices

4.1 Introduction

The controllability criteria discussed earlier are typically inapplicable for many situations because of the complicated structures and tremendous computing overheads of large-scale networked systems. Various measures have been devised for the controllability of complex networks, where most are derived under the premise that the dimension of each node is one(Liu et al., 2011; Lou and Hong, 2012; Nabi-Abdolyousefi and Mesbahi, 2013). However, in real-world networks, nodes frequently have higher-dimensional states that are connected by multidimensional channels(Du et al., 2017; Wang et al., 2017a). In such cases, the question of controllability becomes more sophisticated and intricate. Hao et al.(Hao et al., 2018) studied the controllability of homogeneous networked systems where the network topology is directed and weighted and the node systems have higher-dimensional dynamics, with multiple inputs and multiple outputs. When compared to the result obtained by Wang et al.(Wang et al., 2016b) and Xiang et al.(Xiang et al., 2019b), the conditions are more direct and easier to verify as it does not call for the solution of matrix equations. Also, the corresponding conditions are obtained more precisely for networked MIMO systems in several typical topologies, such as cycles, undirected trees, and globally coupled networks. Our objective is to generalize the result known to a class of heterogeneous networked systems where the state matrices can be different in each node, whereas the control input matrices are identical. The obtained results can be used to re-design an uncontrollable system into a controllable one.

The heterogeneous networked system model under discussion is formulated in Section 4.2. Controllability results obtained by Hao et al.(Hao et al., 2018) are discussed in Section 4.3. Section 4.4 contains our main results as well as numerical examples to support the findings. Controllability of heterogeneous networked systems over some special network topologies are also discussed in Section 4.5 and conclusions are given in Section 4.6.

4.2 Problem formulation

Consider a heterogeneous networked linear time-invariant system with N nodes, where the i^{th} node is described by the following differential equation:

$$\dot{x}_i(t) = A_i x_i(t) + \sum_{j=1}^{N} \beta_{ij} H x_j(t) + d_i B u_i(t), \quad i = 1, 2, \ldots, N \tag{4.1}$$

where, $x_i(t) \in \mathbb{R}^n$ is the state vector; $u_i(t) \in \mathbb{R}^m$ is the external control vector; $A_i \in \mathbb{R}^{n \times n}$ is the state matrix of node i; $B \in \mathbb{R}^{n \times m}$ is the control matrix, with $d_i = 1$ if node i is under control, otherwise $d_i = 0$. $\beta_{ij} \in \mathbb{R}$ represents the coupling strength between the nodes i and j with $\beta_{ij} \neq 0$ if there is a communication from node v_j to node v_i, otherwise $\beta_{ij} = 0$, $i, j = 1, 2, \ldots, N$ and $H \in \mathbb{R}^{n \times n}$ is the inner coupling matrix describing the interconnections among the states $x_j, j = 1, 2, \ldots, N$ of the nodes.
Let

$$L = [\beta_{ij}] \in \mathbb{R}^{N \times N} \quad and \quad D = diag\{d_1, d_2, \ldots, d_N\} \tag{4.2}$$

denote the network topology and external input channels of the networked system (4.1), respectively. Denote the whole state of the networked system by $X = \left[x_1^T, \ldots, x_N^T\right]^T$ and the total external control input vector by $U = \left[u_1^T, \ldots, u_N^T\right]^T$.

Now, using the Kronecker product notation, the networked system (4.1) can be reduced into the following compact form:

$$\dot{X}(t) = \Omega X(t) + \Psi U(t) \tag{4.3}$$

where,

$$\begin{aligned} \Omega &= \mathcal{A} + L \otimes H \\ \Psi &= D \otimes B \end{aligned} \tag{4.4}$$

and $\mathcal{A} = blockdiag\{A_1, A_2, \ldots, A_N\}$. If the state node matrices $A_1, A_2, \ldots, A_N$ are identical, that is, $A_i = A,\ i = 1, 2, \ldots, N$, then the system (4.1) becomes a homogeneous networked system.

4.3 Controllability of Homogeneous Networked Systems over Diagonalizable Network Topology

In this section, we discuss the controllability results obtained by Hao et al.(Hao et al., 2018). The idea of Hao et al.(Hao et al., 2018) was to identify the eigenvalues and eigenvectors of the state matrix Ω and then use PBH eigenvector test to derive necessary and sufficient conditions for the controllability of a homogeneous networked system. Consider the following theorem.

Theorem 4.1. *(Hao et al., 2018) Assume that L is diagonalizable with the set of the eigenvalues $\sigma(L) = \{\lambda_1, \lambda_2, \ldots, \lambda_N\}$. Let $M_i = \{\mu_i^1, \mu_i^2, \ldots, \mu_i^{q_i}\}$ be the set of the eigenvalues of $A + \lambda_i H,\ i = 1, 2, \ldots, N$. Then*

$$\sigma(\Omega) = \left\{\mu_1^1, \mu_1^2, \ldots, \mu_1^{q_i}, \ldots, \mu_N^1, \mu_N^2, \ldots, \mu_N^{q_i}\right\}$$

Moreover, the left eigenvectors of Ω associated with μ_i^j are $t_i \otimes \xi_{ij}^1, t_i \otimes \xi_{ij}^2, \ldots, t_i \otimes \xi_{ij}^{\gamma_{ij}}$ where t_i is the left eigenvector of L corresponding to eigenvalue λ_i: $\gamma_{ij} \geq 1$ is the geometric multiplicity of μ_i^j for $A + \lambda_i H$; $\xi_{ij}^k (k = 1, \ldots, \gamma_{ij})$ are the left eigenvectors of $A + \lambda_i H$ corresponding to $\mu_i^j, j = 1, 2, \ldots, q_i, i = 1, 2, \ldots, N$.

Using the above result Hao et al.(Hao et al., 2018) have proved the following necessary and sufficient conditions for controllability of homogeneous networked systems over a diagonalizable network topology.

Theorem 4.2. *(Hao et al., 2018) Consider a homogeneous networked system with a diagonalizable network topology matrix L. Let $\sigma(L) = \{\lambda_1, \lambda_2, \ldots, \lambda_N\}$. Then the networked system* (4.3)-(4.4) *is controllable if and only if the following conditions are satisfied.*

(i) (L, D) *is controllable;*

(ii) $(A + \lambda_i H, B)$ *is controllable, for $i = 1, 2, \cdots, N$; and*

(iii) If matrices $A + \lambda_{i_1} H, \ldots, A + \lambda_{i_p} H\ (\lambda_{i_k} \in \sigma(L),\ for\ k = 1, \ldots, p,\ p > 1)$ have a common eigenvalue ρ, then $(t_{i_1} D) \otimes (\xi_{i_1}^1 B), \ldots, (t_{i_1} D) \otimes (\xi_{i_1}^{\gamma_{i_1}} B), \ldots, (t_{i_p} D) \otimes$

$(\xi_{i_p}^1 B), \ldots, (t_{i_p} D) \otimes (\xi_{i_p}^{\gamma_{i_p}} B)$ are linearly independent, where t_{i_k} is the left eigenvector of L corresponding to the eigenvalue λ_{i_k}; $\gamma_{i_k} \geq 1$ is the geometric multiplicity of ρ for $A + \lambda_{i_k} H$; $\xi_{i_k}^l (l = 1, \ldots, \gamma_{i_k})$ are the left eigenvectors of $A + \lambda_{i_k} H$ corresponding to $\rho, k = 1, \ldots, p$.

4.4 Controllability of Heterogeneous Networked Systems with Identical Control Input Matrices

In this section, we investigate the controllability of (4.1) under certain network topologies. Suppose that the network topology matrix L is triangularizable. That is, there exists a non-singular matrix T such that $TLT^{-1} = J$, where $J = uppertriang\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$ is the Jordan Canonical Form of L. Let $\sigma(A_i + \lambda_i H) = \{\mu_i^1, \ldots, \mu_i^{q_i}\}$ denotes the set of eigenvalues of $A_i + \lambda_i H, i = 1, 2, \ldots, N$ and $\xi_{ij}^k, k = 1, \ldots, \gamma_{ij}$ be the left eigenvectors of $A_i + \lambda_i H$ corresponding to $\mu_i^j, j = 1, \ldots, q_i, i = 1, \ldots, N$, where $\gamma_{ij} \geq 1$ is the geometric multiplicity of the eigenvalue μ_i^j.

We investigate the controllability of the original system (4.1) in terms of the eigenvalues and left eigenvectors of the state matrix Ω in the compact form (4.3). When the network topology matrix L is triangularizable with triangulizing matrix T and if $T \otimes I$ commutes with $\mathcal{A}$, we characterize the eigenvalues and left eigenvectors of Ω in terms of the eigenvalues and left eigenvectors of $A_i + \lambda_i H,\ i = 1, 2, \ldots, N$ as shown in the following theorem.

Theorem 4.3. *(Ajayakumar and George, 2023b) Let T be the triangulizing matrix for the network topology matrix L and suppose $T \otimes I$ commutes with $\mathcal{A}$. Let (μ_i^j, ξ_{ij}^k) denotes the left eigenpair of $A_i + \lambda_i H$. Then the following statements hold true.*

(i) The eigenspectrum of Ω is the union of eigenspectrum of $A_i + \lambda_i H$, where, $i = 1, 2, \ldots, N$. That is,

$$\sigma(\Omega) = \cup_{i=1}^N \sigma(A_i + \lambda_i H) = \{\mu_1^1, \ldots, \mu_1^{q_1}, \ldots, \mu_N^1, \ldots, \mu_N^{q_N}\}$$

(ii) If J is a diagonal matrix, then $e_i T \otimes \xi_{ij}^k, k = 1, \ldots, \gamma_{ij}$ are the left eigenvectors of Ω corresponding to the eigenvalue $\mu_i^j, j = 1, \ldots, q_i, i = 1, \ldots, N$, where $\{e_i : i = 1, 2, \ldots, N\}$ is the canonical basis for $\mathbb{R}^N$.

(iii) If J contains a Jordan block of order $l \geq 2$ for some eigenvalue λ_{i_0} of L with $\xi_{ij}^k H = 0$ for all $i = i_0, i_0 + 1, \ldots, i_0 + l - 1, j = 1, 2, \ldots, q_i, k = 1, 2, \ldots, \gamma_{ij}$, then $e_i T \otimes$

$\xi_{ij}^k, k = 1, \ldots, \gamma_{ij}$ *are the left eigenvectors of* Ω *corresponding to the eigenvalue* $\mu_i^j, i = 1, 2, \ldots, N, j = 1, 2, \ldots, q_i$.

Proof. (i) By hypothesis, T is a non-singular matrix such that $TLT^{-1} = J$, where $J = uppertriang\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$ is the Jordan Canonical form of L. We will define a matrix $\tilde{\Omega}$ similar to Ω so that we can characterize the eigenvalues of Ω using the eigenvalues of $\tilde{\Omega}$. Also, we will compute the eigenvectors of Ω using the eigenvectors of $\tilde{\Omega}$. Define

$$\tilde{\Omega} = (T \otimes I)\Omega(T^{-1} \otimes I) = (T \otimes I)(\mathcal{A} + L \otimes H)(T^{-1} \otimes I)$$

As $T \otimes I$ commutes with A, we have

$$\begin{aligned}
\tilde{\Omega} &= \mathcal{A}(T \otimes I)(T^{-1} \otimes I) + (T \otimes I)(L \otimes H)(T^{-1} \otimes I) \text{ (using Theorem 2.3)} \\
&= \mathcal{A} + (TLT^{-1} \otimes H) \\
&= \mathcal{A} + J \otimes H \\
&= \mathcal{A} + uppertriang\{\lambda_1, \lambda_2, \ldots, \lambda_N\} \otimes H \\
&= blockuppertriang\{A_1 + \lambda_1 H, \cdots, A_N + \lambda_N H\}
\end{aligned}$$

As the eigenvalues of a block upper triangular matrix are the union of the eigenvalues of the matrices on the diagonal blocks, we have

$$\sigma(\tilde{\Omega}) = \cup_{i=1}^N \sigma(A_i + \lambda_i H) = \{\mu_1^1, \ldots, \mu_1^{q_1}, \ldots, \mu_N^1, \ldots, \mu_N^{q_N}\}$$

By Theorem 2.1, similar matrices have same eigenvalues. Thus, both $\tilde{\Omega}$ and Ω have same eigenvalues and hence

$$\sigma(\Omega) = \cup_{i=1}^N \sigma(A_i + \lambda_i H) = \{\mu_1^1, \ldots, \mu_1^{q_1}, \ldots, \mu_N^1, \ldots, \mu_N^{q_N}\}$$

(ii) Let $\xi_{ij}^k, k = 1, \ldots, \gamma_{ij}$ be the left eigenvectors of $A_i + \lambda_i H$ corresponding to $\mu_i^j, j = 1, \ldots, q_i, i = 1, \ldots, N$. If J is a diagonal matrix, then $\tilde{\Omega}$ is a block diagonal matrix and hence $e_i \otimes \xi_{ij}^k, k = 1, \ldots, \gamma_{ij}$ are left eigenvectors of $\tilde{\Omega}$ corresponding to $\mu_i^j, j = 1, \ldots, q_i, i = 1, \ldots, N$. Then, by Theorem 2.1

$$\left(e_i \otimes \xi_{ij}^k\right)(T \otimes I) = e_i T \otimes \xi_{ij}^k, k = 1, \ldots, \gamma_{ij}$$

are the left eigenvectors of Ω corresponding to the eigenvalue $\mu_i^j, j = 1, \ldots, q_i, i =$

$1, \ldots, N$, where $\{e_i : i = 1, 2, \ldots, N\}$ is the canonical basis for $\mathbb{R}^N$.

(iii) Suppose that J contains a Jordan block of order 2, corresponding to the eigenvalue λ_{i_0} of L. Then the matrix $\tilde{\Omega}$ contains the block matrix of the form

$$\mathcal{J} = \begin{bmatrix} A_{i_0} + \lambda_{i_0} H & H \\ 0 & A_{i_0+1} + \lambda_{i_0+1} H \end{bmatrix} \tag{4.5}$$

It follows easily that, $e_{i_0+1} \otimes \xi^k_{i_0+1j}, k = 1, 2, \ldots, \gamma_{i_0+1j}$ are eigenvectors of $\tilde{\Omega}$ corresponding to the eigenvalues $\mu^j_{i_0+1}, j = 1, 2, \ldots, q_{i_0+1}$. If $\xi^k_{i_0 j_0} H = 0$ for all $k = 1, 2, \ldots, \gamma_{i_0 j_0}$, then $e_{i_0} \otimes \xi^k_{i_0 j_0}, k = 1, 2, \ldots, \gamma_{i_0 j_0}$ are left eigenvectors of $\tilde{\Omega}$ corresponding to the eigenvalue $\mu^{j_0}_{i_0}$. Now suppose that J contains a Jordan block of order $l \geq 2$ for some eigenvalue λ_{i_0} of L, then again we can consider $(l-1)$ block matrices of the form (4.5) and by using the fact that $\xi^k_{ij} H = 0$ for all $i = i_0, i_0+1, \ldots, i_0+l-1, j = 1, 2, \ldots, q_i, k = 1, 2, \ldots, \gamma_{ij}$ we get $e_i \otimes \xi^k_{ij}, k = 1, 2, \ldots, \gamma_{ij}$ are left eigenvectors of $\tilde{\Omega}$ corresponding to the eigenvalue $\mu^j_i, i = 1, 2, \ldots, N, j = 1, 2, \ldots, q_i$.

We will prove that these are the only eigenvectors of $\tilde{\Omega}$. Suppose that $\tilde{\Omega}$ does not have any Jordan blocks and let $\xi = \begin{bmatrix} \xi_1 & \xi_2 & \ldots & \xi_N \end{bmatrix} \in \mathbb{R}^{Nn}$ be a left eigenvector of $\tilde{\Omega}$ corresponding to the eigenvalue μ, where $\xi_1, \xi_2, \ldots, \xi_N \in \mathbb{R}^n$. Then $\xi^T \tilde{\Omega} = \mu \xi^T$ implies that

$$\begin{bmatrix} \xi_1 (A_1 + \lambda_1 H) \\ \xi_2 (A_2 + \lambda_2 H) \\ \vdots \\ \xi_N (A_N + \lambda_N H) \end{bmatrix}^T = \mu \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_N \end{bmatrix}^T$$

This in-turn implies that μ is an eigenvalue of $A_i + \lambda_i H$ for all i with ξ_i as an eigenvector. Suppose that $\tilde{\Omega}$ has a block of type (4.5). Then $\xi^T \tilde{\Omega} = \mu \xi^T$ implies that

$$\begin{bmatrix} \xi_1 (A_1 + \lambda_1 H) \\ \vdots \\ \xi_i (A_i + \lambda_i H) \\ \xi_i H + \xi_{i+1} H (A_2 + \lambda_2 H) \\ \vdots \\ \xi_1 (A_N + \lambda_N H) \end{bmatrix}^T = \mu \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_i \\ \xi_{i+1} \\ \vdots \\ \xi_N \end{bmatrix}^T$$

As $\xi_i (A_i + \lambda_i H) = \mu \xi_i$, ξ_i is a left eigenvector of $A_i + \lambda_i H$. Then by our hypothesis,

$\xi_i H = 0$. Hence μ is an eigenvalue of $A_i + \lambda_i H$ for all i with ξ as an eigenvector. Thus, if $A_i + \lambda_i H, i = 1, 2, \ldots, N$ does not have a common eigenvalue, then the left eigenvectors of $\tilde{\Omega}$ are of the form $e_i \otimes \xi$, where ξ is a left eigenvector of $A_i + \lambda_i H$ for some i. If they have a common eigenvalue, the eigenvectors are either of the form $e_i \otimes \xi$, where ξ is a left eigenvector of $A_i + \lambda_i H$ for some i or of the form $\sum_{\alpha=1}^{r} e_{i_\alpha} \otimes \xi_{i_\alpha}$, where $A_i + \lambda_i H, i \in \{i_1, i_2, \ldots, i_r\}$ have a common eigenvalue μ with eigenvector ξ_{i_α} for each $i_1, i_2, \ldots, i_r$.

Thus in both cases, $\left(e_i \otimes \xi_{ij}^k\right)(T \otimes I) = e_i T \otimes \xi_{ij}^k (k = 1, \ldots, \gamma_{ij})$ are the left eigenvectors of Ω corresponding to $\mu_i^j, j = 1, \ldots, q_i, i = 1, \ldots, N$.

□

Using the above result, we will prove the following necessary and sufficient conditions for controllability of the heterogeneous networked system (4.3).

Theorem 4.4. *(Ajayakumar and George, 2023b) Let T be a non-singular matrix triagularizing matrix L such that $T \otimes I$ commutes with $\mathcal{A}$. If J contains a Jordan block of order $l \geq 2$ corresponding to the eigenvalue λ_{i_0} of L, then assume that $\xi_{ij}^k H = 0$ for all $i = i_0, i_0 + 1, \ldots, i_0 + l - 1, j = 1, 2, \ldots, q_i, k = 1, 2, \ldots, \gamma_{ij}$, where $\xi_{ij}^k, i = 1, 2, \ldots, N, j = 1, 2, \ldots, q_i, k = 1, 2, \ldots, \gamma_{ij}$ are the left eigenvectors of $A_i + \lambda_i H$ corresponding to the eigenvalues $\mu_i^j, i = 1, 2, \ldots, N, j = 1, 2, \ldots, q_i$. Then the networked system* (4.3) *is controllable if and only if*

(i) $e_i T D \neq 0$ *for all* $i = 1, \ldots, N$

(ii) $(A_i + \lambda_i H, B)$ *is controllable, for* $i = 1, 2, \cdots, N$*; and*

(iii) If matrices $A_{i_1} + \lambda_{i_1} H, A_{i_2} + \lambda_{i_2} H, \ldots, A_{i_p} + \lambda_{i_p} H (\lambda_{i_k} \in \sigma(L), k = 1, \ldots, p,$ where $p > 1)$ have a common eigenvalue ρ, then $(e_{i_1} T D) \otimes (\xi_{i_1}^1 B), \cdots, (e_{i_1} T D) \otimes (\xi_{i_1}^{\gamma_{i_1}} B), \ldots, (e_{i_p} T D) \otimes (\xi_{i_p}^1 B), \ldots, (e_{i_p} T D) \otimes (\xi_{i_p}^{\gamma_{i_p}} B)$ are linearly independent vectors, where $\gamma_{i_k} \geq 1$ is the geometric multiplicity of σ for $A_{i_k} + \lambda_{i_k} H$ and $\xi_{i_k}^l (l = 1, \ldots, \gamma_{i_k})$ are the left eigenvectors of $A_{i_k} + \lambda_{i_k} H$ corresponding to $\sigma, k = 1, \ldots, p$.

Proof. **(Necessary part)** From Theorem 4.3 it follows that, $e_i T \otimes \xi_{ij}^k (k = 1, \ldots, \gamma_{ij})$ are left eigenvectors of Ω corresponding to $\mu_i^j, j = 1, \ldots, q_i, i = 1, \ldots, N$. If the networked system (4.3) is controllable, then by PBH eigenvector test

$$(e_i T \otimes \xi_{ij}^l)(D \otimes B) \neq 0, \ for \ l = 1, \ldots, \gamma_{ij}, j = 1, \ldots, q_i, i = 1, \ldots, N$$

which implies that

$$e_iTD \neq 0,\ i = 1, \ldots, N,$$

and

$$\xi_{ij}^l B \neq 0,\ for\ l = 1, \ldots, \gamma_{ij}, j = 1, \ldots, q_i, i = 1, \ldots, N$$

Since ξ_{ij}^l is an arbitrary left eigenvector of $A_i + \lambda_i H$, the controllability of $(A_i + \lambda_i H, B)$, for $i = 1, \ldots, N$ follows.

Assume that the matrices $A_{i_1} + \lambda_{i_1} H, \cdots, A_{i_p} + \lambda_{i_p} H (\lambda_{i_k} \in \sigma(L), k = 1, \ldots, p,$ where $p > 1)$ have a common eigenvalue ρ. Then all the left eigenvectors of Ω corresponding to ρ can be expressed in the form of $\sum_{k=1}^{p} \sum_{l=1}^{\gamma_{i_k}} \alpha_{kl}(e_{i_k}T \otimes \xi_{i_k}^l)$, where $\alpha_{kl} \in \mathbb{R}(k = 1, \ldots, p, l = 1, \ldots, \gamma_{i_k})$ are scalars, not all are zero and $\xi_{i_k}^1, \ldots, \xi_{i_k}^{\gamma_{i_k}}$, are the eigenvectors of $A_{i_k} + \lambda_{i_k} H$ corresponding to the eigenvalue ρ, where $k = 1, \ldots, p$. If the networked system is controllable, then

$$\left[\sum_{k=1}^{p} \sum_{l=1}^{\gamma_{i_k}} \alpha_{kl}(e_{i_k}T \otimes \xi_{i_k}^l)\right](D \otimes B) \neq 0$$

Consequently, we have

$$\sum_{k=1}^{p} \sum_{l=1}^{\gamma_{ik}} \alpha_{kl}(e_{i_k}TD) \otimes (\xi_{ik}^l B) \neq 0$$

for any scalars $\alpha_{kl} \in \mathbb{R}(k = 1, \ldots, p, l = 1, \ldots, \gamma_{ik})$, not all of them are zero. Therefore, $(e_{i1}TD) \otimes (\xi_{i1}^1 B), \ldots, (e_{i1}TD) \otimes (\xi_{i1}^{\gamma_{i1}} B), \ldots, (e_{ip}TD) \otimes (\xi_{ip}^1 B), \ldots, (e_{ip}TD) \otimes (\xi_{ip}^{\gamma_{ip}} B)$ are linearly independent vectors in $\mathbb{R}^{Nn}$.

(Sufficiency part) Suppose that the networked system is uncontrollable, then we will prove that at least one condition in Theorem 4.4 does not hold. If the networked system is not controllable, then there exists a left eigenpair of Ω, denoted as $(\tilde{\mu}, \tilde{v})$, such that $\tilde{v}\Psi = 0$.

If $\tilde{\mu} \in \sigma(A_{i_0} + \lambda_{i_0} H)$ and $\tilde{\mu} \notin \sigma(A_1 + \lambda_1 H) \cup \ldots \cup \sigma(A_{i_0-1} + \lambda_{i_0-1} H) \cup \sigma(A_{i_0+1} + \lambda_{i_0+1} H) \cup \ldots \cup \sigma(A_{i_N} + \lambda_{i_N} H)$. Again $\tilde{v}$ can be written as a linear combination, $\tilde{v} = \sum_{l=1}^{\gamma_{i_0 j_0}} \alpha_0^l (e_{i_0}T \otimes \xi_{i_{0 j_0}}^l)$, where $\xi_{i_{0 j_0}}^1, \ldots, \xi_{i_{0 j_0}}^{\gamma_{i_0 j_0}}$ of left eigenvectors of $A_{i_0} + \lambda_{i_0} H$ corresponding to $\tilde{\mu}$, where, $\left[\alpha_0^1, \ldots, \alpha_0^{\gamma_{i_0 j_0}}\right]$ is some non-zero vector. Now $\tilde{v}\Psi = 0$ implies

$$\begin{aligned}
\sum_{l=1}^{\gamma_{i_0 j_0}} \alpha_0^l (e_{i_0}T \otimes \xi_{i_{0 j_0}}^l)(D \otimes B) &= \sum_{l=1}^{\gamma_{i_0 j_0}} \alpha_0^l (e_{i_0}TD) \otimes (\xi_{i_{0 j_0}}^l B) \\
&= (e_{i_0}TD) \otimes \left(\sum_{l=1}^{\gamma_{i_0 j_0}} \alpha_0^l \xi_{i_{0 j_0}}^l B\right) = 0
\end{aligned}$$

This implies that $e_{i_0}TD = 0$ or $\sum_{l=1}^{\gamma_{i_0 j_0}} \alpha_0^l \xi_{i_{0 j_0}}^l B = 0$. If $\sum_{l=1}^{\gamma_{i_0 j_0}} \alpha_0^l \xi_{i_{0 j_0}}^l B = 0$, then $(A_{i_0} + \lambda_{i_0} H, B)$ is uncontrollable as $\sum_{l=1}^{\gamma_{i_0 j_0}} \alpha_0^l \xi_{i_{0 j_0}}^l$ is a left eigenvector of $A_{i_0} + \lambda_{i_0} H$. Thus, if the networked system is uncontrollable, then either there exists $\lambda_{i_0} \in \sigma(L)$ such that $(A_{i_0} + \lambda_{i_0} H, B)$ is uncontrollable or $e_{i_0}TD = 0$ for some i_0.

Let ρ be the common eigenvalue of the matrices $A_{i_1} + \lambda_{i_1} H, \dots, A_{i_p} + \lambda_{i_p} H(\lambda_{i_k} \in \sigma(L),\ for\ k = 1, \dots, p, p > 1)$ and the corresponding eigenvectors of $A_{i_k} + \lambda_{i_k}$ are $\xi_{i_k}^1, \dots, \xi_{i_k}^{\gamma_{i_k}}$, where $k = 1, \dots, p$. Since $\tilde{v}$ can be expressed in the form $\sum_{k=1}^{p} \sum_{l=1}^{\gamma_{i_k}} \alpha_0^{kl} \left(e_{i_k} T \otimes \xi_{ik}^l\right)$, where $\alpha_0^{kl} (l = 1, \dots, \gamma_{i_k}, k = 1, \dots, p)$ are some scalars, which are not all zero. Then $\tilde{v}\Psi = 0$ implies that there exists a non-zero vector $\left[\alpha_0^{11}, \dots, \alpha_0^{1\gamma_{i_1}}, \dots, \alpha_0^{p1}, \dots, \alpha_0^{p\gamma_{i_p}}\right]$ such that

$$\left[\sum_{k=1}^{p} \sum_{l=1}^{\gamma_{i_k}} \alpha_0^{kl} \left(e_{i_k} T \otimes \xi_{i_k}^l\right)\right] (D \otimes B) = \sum_{k=1}^{p} \sum_{l=1}^{\gamma_{i_k}} \alpha_0^{kl} \left[(e_{i_k} TD) \otimes (\xi_{i_k}^l B)\right] = 0$$

This implies that $(e_{i_1}TD) \otimes (\xi_{i_1}^1 B), \dots, (e_{i_1}TD) \otimes (\xi_{i_1}^{\gamma_{i_1}} B), \dots, (e_{i_p}TD) \otimes (\xi_{i_p}^1 B), \dots, (e_{i_p}TD) \otimes (\xi_{i_p}^{\gamma_{i_p}} B)$ are linearly dependent.

Therefore, if the networked system is uncontrollable, then at least one condition in Theorem 4.4 does not hold, true. □

The following examples demonstrate the application of the result for testing controllability of heterogeneous networked systems.

Example 4.1. (Ajayakumar and George, 2023b) Consider a heterogeneous networked system as shown in Figure 4.1 composed of 3 nodes in which two nodes are identical. The state matrices of each node (A_1, A_2, A_3), control matrix B and inner coupling matrix H are given by

$$A_1 = A_3 = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \ H = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \tag{4.6}$$

The network topology matrix and the external input channel matrix are given by

$$L = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

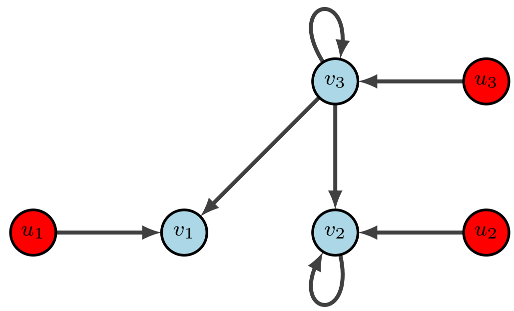

Figure 4.1: Controllable heterogeneous networked system with triangularizable network topology L and node dynamics as given in (4.6).

For the network topology matrix L, there exists a non-singular matrix $T = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ such that

$$TLT^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = J$$

Clearly, $T \otimes I$ commutes with $\mathcal{A}$. The eigenvalues of L are $\lambda_1 = 0, \lambda_2 = 1$ and $\lambda_3 = 1$ and J contains a Jordan block of order 2 corresponding to the eigenvalue $\lambda_2 = 1$. Thus, we have to verify whether the left eigenvectors of $A_2 + H = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ are orthogonal to the columns of H. Observe that the eigenvalues of the matrix $A_2 + H$ are 1,1,1 and $\xi_{21}^1 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$ is the only left eigenvector of $A_2 + H$ corresponding to the eigenvalue 1. Also, it satisfies $\xi_{21}^1 H = 0$. Then, we can easily verify the following:

(i) As $TD = T = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $e_i TD \neq 0$ for all $i = 1, 2, 3$.

(ii) $(A_1, B), (A_2 + H, B)$ and $(A_3 + H, B)$ are controllable.

(iii) $\rho = 1$ is a common eigenvalue of the matrices $A_2 + H$ and $A_3 + H$ have with left eigenvectors $\xi_{21}^1 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$ and $\xi_{31}^1 = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}$, respectively. Also, the vectors

$$e_2 TD \otimes \xi_{21}^1 B = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$

and

$$e_3TD \otimes \xi_{31}^1 B = \begin{bmatrix} 0 & 0 & -1 \end{bmatrix}$$

are linearly independent vectors.

As all the conditions $(i) - (iii)$ of Theorem 4.4 are verified, the heterogeneous networked system is controllable. The controllability of the given system by also using Kalman's rank condition. The system can be written in the compact form (4.3), with

$$\Omega = \begin{bmatrix} 1 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \text{ and } \Psi = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

We can see that the controllability matrix $\mathcal{Q}(\Omega, \Psi)$ has rank 9 and hence the given system is controllable.

Example 4.2. (Ajayakumar and George, 2023b) Consider a heterogeneous networked system shown in Figure 4.2, which is composed of 3 nodes in which two nodes are identical. The state matrices of each node (A_1, A_2, A_3), control matrix B and inner coupling matrix H are given by

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & -1 \end{bmatrix}, A_2 = A_3 = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & 2 & -1 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \; H = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \tag{4.7}$$

The network topology matrix and the external input channel matrix are given by

$$L = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

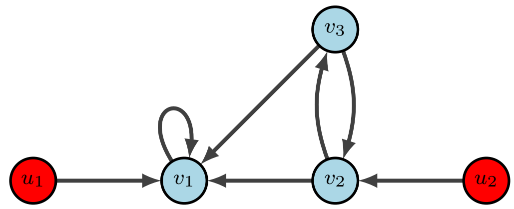

Figure 4.2: Controllable heterogeneous networked system with triangularizable network topology L and node dynamics as in (4.7).

For the network topology matrix L, there exists a triangularizing non-singular matrix $T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$ such that

$$TLT^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 1 \\ 0 & \frac{1}{2} & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = J$$

Clearly, $T \otimes I$ commutes with $\mathcal{A}$. The eigenvalues of L are, $\lambda_1 = 1, \lambda_2 = 1$ and $\lambda_3 = -1$. Also, J contains a Jordan block of order 2 corresponding to the eigenvalue 1. Observe that the eigenvalues of the matrix $A_1 + H = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & -1 \\ 0 & 1 & 0 \end{bmatrix}$ are 1,1,1 and $\xi_{11}^1 = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}$ is the only left eigenvector of the matrix $A_1 + H$ corresponding to the eigenvalue 1. Also, $\xi_{11}^1 H = 0$. Further, we can verify that

(i) $e_i TD \neq 0$ for all $i = 1, 2, 3$.

(ii) $(A_1 + H, B),\ (A_2 + H, B)$ and $(A_3 - H, B)$ are controllable.

(iii) As the matrices $A_1 + H$,$A_2 + H$ and $A_3 - H$ do not have a common eigenvalue, the condition (iii) in Theorem 4.4 is satisfied.

Thus all the conditions $(i) - (iii)$ of Theorem 4.4 are verified. Hence, the heterogeneous system is controllable. We can verify the controllability of the given system by also using

Kalman's rank condition. The system can be written in the compact form (4.3), with

$$\Omega = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 2 & -1 \end{bmatrix} \text{ and } \Psi = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We can see that the controllability matrix $\mathcal{Q}(\Omega, \Psi)$ has rank 9 and hence the given system is controllable.

Hao et al.'s(Hao et al., 2018) result cannot be used to verify the controllability of the systems in the above examples as the networked system follows heterogeneous dynamics and the network topology matrix is non-diagonalizble. Now, we consider another example of a controllable networked system having heterogeneous dynamics with diagonalizable network topology matrix.

Example 4.3. (Ajayakumar and George, 2023b) Consider a heterogeneous networked system composed of 3 nodes in which two nodes are identical. The state matrices of each node (A_1, A_2, A_3), control matrix B and inner coupling matrix H are given by

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, A_2 = A_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \ H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The network topology matrix and the external input channel matrix are given by

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

There exists a non-singular matrix $T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & \sqrt{2} & 0 \end{bmatrix}$ such that $TLT^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = J.$

J has no Jordan block of order ≥ 2 and $T \otimes I$ commutes with $\mathcal{A}$. $\lambda_1 = 1, \lambda_2 = 0$ and $\lambda_3 = 1$ are the eigenvalues of L. Also,

(i) $e_i TD \neq 0$ for all $i = 1, 2, 3$.

(ii) $(A_1 + H, B),\ (A_2, B)$ and $(A_3 + H, B)$ are controllable.

(iii) The matrices $A_1 + H$ and $A_3 + H$ have a common eigenvalue 1 with left eigenvectors $\xi_{11}^1 = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}$ and $\xi_{31}^1 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$ respectively. Further,

$$e_1 TD \otimes \xi_{11}^1 B = \begin{bmatrix} -1 & 0 & 0 \end{bmatrix}$$

and

$$e_3 TD \otimes \xi_{31}^1 B = \begin{bmatrix} 0 & \sqrt{2} & 0 \end{bmatrix}$$

are linearly independent vectors.

Thus, all the conditions $(i) - (iii)$ of Theorem 4.4 are verified. Hence, the heterogeneous network system is controllable. We can also verify the controllability of the given system using Kalman's rank condition as the given system can be written in the compact form (4.3), with

$$\Omega = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \text{ and } \Psi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This approach enable as to find the nodes to which a control can be applied to make an uncontrollable system to a controllable system.

Remark 4.1. (Ajayakumar and George, 2023b) If $e_i TD = 0$ for some $i = 1, 2, \ldots, N$, then the given system is not controllable. For, we have, $e_i T \otimes \xi_{ij}^k (k = 1, \cdots, \gamma_{ij})$ are left eigenvectors of Ω corresponding to $\mu_i^j, j = 1, \ldots, q_i, i = 1, \ldots, N$. If $e_i TD = 0$ for some i, say i_0, then $(e_{i_0} T \otimes \xi_{i_0 j}^k)(D \otimes B) = (e_{i_0} TD \otimes \xi_{i_0 j}^k B) = 0$ for all $j = 1, 2, \ldots, q_{i_0}, k = 1, 2, \ldots, \gamma_{i_0 j}$. Then by PBH eigenvector test, the given system is not controllable.

Now we may be able to modify the external input matrix D, so that $e_iTD \neq 0, i = 1, \ldots, N$ as shown in the following example.

Example 4.4. (Ajayakumar and George, 2023b) Consider a homogeneous network system composed of 3 nodes. The state matrices of each node (A_1, A_2, A_3), control matrix B and inner coupling matrix H are given by

$$A_1 = A_2 = A_3 = A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.8}$$

The network topology matrix and the external input channel matrix are given by

$$L = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

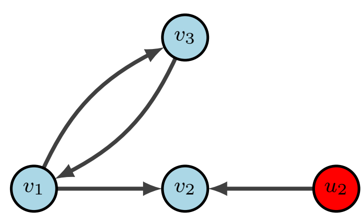

Figure 4.3: Heterogeneous networked system which is not controllable with a triangularazible network topology L and node dynamics given in (4.8).

There exists a non-singular matrix $T = \begin{bmatrix} 0 & 1 & -1 \\ \frac{\sqrt{3}}{2} & 0 & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & 0 & \frac{\sqrt{3}}{2} \end{bmatrix}$ such that $TLT^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = J$. Clearly, $T \otimes I$ commutes with $\mathcal{A}$. From Remark 4.1, it is easy to verify that the networked system is not controllable as

$$e_2TD = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Observe that either d_1 or d_3 must be 1 so that $e_iTD \neq 0$ for all $i = 1, 2, 3$. Modify D as

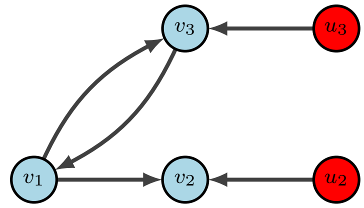

Figure 4.4: The networked system becomes controllable with node dynamics as in (4.8), if the external control input matrix is $\tilde{D}$.

$\tilde{D} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. In other words, either node v_1 or node v_3 is supplied with a control input. Then $e_i T \tilde{D} \neq 0$ for all $i = 1, 2, 3$. For the modified network system, we can verify the conditions (ii) and (iii) of Theorem 4.4. The eigenvalues of L are $\lambda_1 = 0, \lambda_2 = 1$ and $\lambda_3 = -1$. Clearly, $(A, B), (A + H, B), (A - H, B)$ are controllable and these matrices do not have a common eigenvalue. Thus, all the conditions of Theorem 4.4 are satisfied and hence the modified heterogeneous system is controllable.

The condition that the matrix $T \otimes I$ commutes with $\mathcal{A}$ in Theorem 4.4 is automatically satisfied when the networked system is homogeneous as we can see in the following proposition.

Proposition 4.1. *(Ajayakumar and George, 2023b) If the networked system* (4.1) *is a homogeneous system, that is, $A_i = A$ for $i = 1, 2, \dots, N$, then $T \otimes I$ commutes with $\mathcal{A}$.*

Proof. The compact form of the networked system (4.1) is

$$\dot{X}(t) = \Omega X(t) + \Psi U(t)$$

where,

$$\Omega = \mathcal{A} + L \otimes H$$
$$\Psi = D \otimes B$$

If the networked system (4.1) is a homogeneous system, then

$$\mathcal{A} = blockdiag\{A_1, A_2, \dots, A_N\} = blockdiag\{A, A, \dots, A\} = I \otimes A$$

Clearly,

$$\begin{aligned}(T\otimes I)\mathcal{A} &= (T\otimes I)(I\otimes A)\\ &= T\otimes A\\ &= (I\otimes A)(T\otimes I)\\ &= \mathcal{A}(T\otimes I)\end{aligned}$$

Thus, $T\otimes I$ commutes with $\mathcal{A}$. □

Consequently, for a homogeneous networked system, we have the following result.

Theorem 4.5. *(Ajayakumar and George, 2023b) Suppose that the networked system* (4.3) *is a homogeneous system, that is,* $A_i = A$ *for all* $i = 1,\ldots,N$ *with*

(a) a triangularizable network topology. That is, $TLT^{-1} = J = uppertriang\{\lambda_1,\ldots,\lambda_N\}$, *where* J *is the Jordan Canonical Form of* L*; and*

(b) if J *contains a Jordan block of order* $l \geq 2$ *corresponding to the eigenvalue* λ_{i_0} *of* L *and* $\xi_{ij}^k H = 0$ *for all* $i = i_0, i_0+1,\ldots,i_0+l-1, j = 1,2,\ldots,q_i, k = 1,2,\ldots,\gamma_{ij}$, *where* $\xi_{ij}^k, i = 1,2,\ldots,k = 1,2,\ldots,\gamma_{ij}$ *are the left eigenvectors of* $A+\lambda_i H$ *corresponding to the eigenvalues* μ_i^j *and* $\gamma_{ij} \geq 1$ *represents the geometric multiplicity of* μ_i^j.

Then the networked system (4.3) *is controllable if and only if the following conditions are satisfied.*

(i) $e_i TD \neq 0$ *for all* $i = 1,\ldots,N$ *, where* $\{e_i\}$ *is the canonical basis for* $\mathbb{R}^N$.

(ii) $(A+\lambda_i H, B)$ *is controllable, for* $i = 1,2,\cdots,N$*; and*

(iii) *If matrices* $A+\lambda_{i_1}H,\ldots,A+\lambda_{i_p}H(\lambda_{i_k}\in\sigma(L),\ for\ k=1,\ldots,p,\ p>1)$ *have a common eigenvalue* ρ, *then* $(e_{i_1}TD)\otimes(\xi_{i_1}^1 B),\ldots,(e_{i_1}TD)\otimes(\xi_{i_1}^{\gamma_{i_1}}B),\ldots,(e_{i_p}TD)\otimes(\xi_{i_p}^1 B),\ldots,(e_{i_p}TD)\otimes(\xi_{i_p}^{\gamma_{i_p}}B)$ *are linearly independent vectors where* $\gamma_{i_k}\geq 1$ *is the geometric multiplicity of the eigenvalue* ρ *for the matrix* $A+\lambda_{i_k}H$ *and* $\xi_{i_k}^l(l = 1,\ldots,\gamma_{i_k})$ *are the left eigenvectors of* $A+\lambda_{i_k}H$ *corresponding to* $\sigma, k = 1,\ldots,p$.

In the following example, we verify the conditions of $(i)-(iii)$ Theorem 4.5 to obtain the controllability of a homogeneous networked system.

Example 4.5. (Ajayakumar and George, 2023b) Consider a networked system with two identical nodes. The state matrix A, control matrix B and inner coupling matrix H are given by

$$A_1 = A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, H = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

The network topology matrix and the external input channel matrix are given by

$$L = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Then, there exists a non-singular matrix $T = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ such that $TLT^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. Here, $\lambda_1 = -1$ and $\lambda_2 = 1$ are the eigenvalues of L. Observe that

(i) $e_i TD \neq 0$ for all $i = 1, 2$.

(ii) $(A_1 - H, B), (A_2 + H, B)$ are controllable. As the matrices $A_1 - H$ and $A_2 + H$ do not have a common eigenvalue, condition (iii) is automatically satisfied.

Thus, all the conditions of Theorem 4.5 are verified. Hence, the homogeneous networked system is controllable.

Remark 4.2. (Ajayakumar and George, 2023b) Verification of the following conditions restrict the application of Theorem 4.4 to a general heterogeneous networked system.

(i) $T \otimes I$ commutes with A.

(ii) If the Jordan canonical form J of the network topology matrix L contains a Jordan block of order $l \geq 2$ corresponding to the eigenvalue λ_{i_0} of L, then assume that $\xi_{ij}^k H = 0$ for all $i = i_0, i_0 + 1, \ldots, i_0 + l - 1, j = 1, 2, \ldots, q_i, k = 1, 2, \ldots, \gamma_{ij}$, where $\xi_{ij}^k, i = 1, 2, \ldots, N, j = 1, 2, \ldots, q_i, k = 1, 2, \ldots, \gamma_{ij}$ are the left eigenvectors of $A_i + \lambda_i H$ corresponding to the eigenvalues $\mu_i^j, i = 1, 2, \ldots, N, j = 1, 2, \ldots, q_i$.

However, condition (i) is trivially satisfied in the case for a homogeneous networked system and condition (ii) is automatically satisfied when the network topology is diagonalizable. The network topology being triangularizable is an advantage over the existing results as the available results are only for systems with a diagonalizable network topology. If a triangularizable network topology is applied to a homogeneous system, Hao et al.'s(Hao et al., 2018) result does not ensure controllability of the system as the network topology matrix is not diagonalizable. But, we have seen in Example 4.4 that our result can be applied to a homogeneous networked system with non-diagonalizable network topology. Also, as seen in Examples 4.1 - 4.3, our result can be applied to heterogeneous networked systems with triangularizable network topology matrix. From the given examples it is evident that our result is applicable to a larger class of networked systems. Another advantage is that, as shown in Example 4.4, we can identify nodes of an uncontrollable system in which one can apply control to a node to make the modified networked system controllable. Thus, the results have application in design of network topology.

Hao et al. (Hao et al., 2018) have proved Theorem 4.1 as a necessary and sufficient condition for the controllability of a homogeneous networked system with a diagonalizable network topology matrix. With the help of the following proposition, we now show that Theorem 4.4 is a generalization of Theorem 4.1 of Hao et al.(Hao et al., 2018).

Proposition 4.2. *(Ajayakumar and George, 2023b) Suppose that the network topology matrix L is diagonalizable. That is, there exists a matrix T such that $TLT^{-1} = J$, where $J = diag\{\lambda_1, \dots, \lambda_N\}$. Then (L, D) is controllable if and only if $e_iTD \neq 0, i = 1, 2, \dots, N$.*

Proof. Given that there exists a matrix T such that $TLT^{-1} = J$, where $J = diag\{\lambda_1, \dots, \lambda_N\}$. Now,

$$\begin{aligned} TLT^{-1} = J \Rightarrow TL = JT& \\ &\Rightarrow e_iTL = e_iJT \; \forall \; i = 1, 2, \dots, N \\ &\Rightarrow (e_iT)L = \lambda_i(e_iT) \; \forall \; i = 1, 2, \dots, N \end{aligned}$$

That is, e_iT is a left eigenvector of L corresponding to the eigenvalue $\lambda_i, i = 1, 2, \dots, N$. Then by PBH eigenvector test, (L, D) is controllable if and only if $e_iTD \neq 0, i = 1, 2, \dots, N$. □

Thus by Proposition 4.2, we can now deduce the necessary and sufficient condition for the controllability of a homogeneous networked system with a diagonalizable network topology matrix L, established by Hao et al.(Hao et al., 2018) as a corollary of Theorem 4.5 as follows.

Corollary 4.1. *(Hao et al., 2018) Consider a homogeneous networked system, that is, $A_i = A$ for all $i = 1, \dots, N$ with a diagonalizable network topology matrix L. Let $\sigma(L) = \{\lambda_1, \lambda_2, \dots, \lambda_N\}$. Then the networked system* (4.1) *is controllable if and only if the following conditions are satisfied:*

(i) (L, D) *is controllable;*

(ii) $(A + \lambda_i H, B)$ *is controllable, for* $i = 1, 2, \cdots, N$*; and*

(iii) *If matrices* $A + \lambda_{i_1} H, \ldots, A + \lambda_{i_p} H$ $(\lambda_{i_k} \in \sigma(C),\ for\ k = 1, \ldots, p,\ p > 1)$ *have a common eigenvalue* ρ*, then* $(t_{i_1} D) \otimes (\xi_{i_1}^1 B), \ldots, (t_{i_1} D) \otimes (\xi_{i_1}^{\gamma_{i_1}} B), \ldots, (t_{i_p} D) \otimes (\xi_{i_p}^1 B), \ldots, (t_{i_p} D) \otimes (\xi_{i_p}^{\gamma_{i_p}} B)$ *are linearly independent, where* t_{i_k} *is the left eigenvector of* L *corresponding to the eigenvalue* λ_{i_k}*;* $\gamma_{i_k} \geq 1$ *is the geometric multiplicity of the eigenvalue* ρ *for the matrix* $A + \lambda_{i_k} H$*;* $\xi_{i_k}^l (l = 1, \ldots, \gamma_{i_k})$ *are the left eigenvectors of* $A + \lambda_{i_k} H$ *corresponding to* $\rho, k = 1, \ldots, p$*.*

In view of Proposition 4.2, the condition (i) of Theorem 4.5 and condition (i) of Corollary 4.1 are equivalent. The condition (ii) in Theorem 4.5 and Corollary 4.1 coincide. As per the result of Hao et. al (Hao et al., 2018), if (λ_i, t_i) and (μ, ξ) are the left eigenpairs of L and $A + \lambda_i H$, respectively, then $(\mu, \xi(t_i \otimes I_n))$ is a left eigenpair of $\Omega = I_N \otimes \tilde{A} + L \otimes H$. This inturn implies that, the condition (iii) in Theorem 4.5 is equivalent to condition (iii) in Corollary 4.1.

Remark 4.3. The existence of the matrix T satisfying all the required conditions is crucial in applying the theorem. If the given system is such that $A_i \neq A_j$ for all $i \neq j$, then for $\mathcal{A} = blockdiag\{A_1, A_2, \ldots, A_N\}$ to commute with $(T \otimes I)$, T must be a diagonal matrix. If $A_i = A_j$ for some $i \neq j$, then T_{ij} and T_{ji} are the only possible non-zero elements besides the diagonal entries.

4.5 Controllability of Heterogeneous Systems over Specific Network Topologies

Now we investigate controllability properties of networked systems with some specific network topologies. In a networked system, if there exists a node v_j having no incoming edge, we obtain a necessary condition for controllability as shown below.

Theorem 4.6. *(Ajayakumar and George, 2023b) Suppose that there exists a node* v_j *with no edge from any other nodes. If* (A_j, B) *is not controllable, then the networked system* (4.3) *is not controllable.*

Proof. If there exists a node v_j with no edge from any other nodes, the network topology matrix L is of the form

$$L = \begin{bmatrix} \beta_{11} & \beta_{12} & \dots & \beta_{1N} \\ \beta_{21} & \beta_{22} & \dots & \beta_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ \beta_{(j-1)1} & \beta_{(j-1)2} & \dots & \beta_{(j-1)N} \\ 0 & 0 & \dots & 0 \\ \beta_{(j+1)1} & \beta_{(j+1)2} & \dots & \beta_{(j+1)N} \\ \vdots & \vdots & \vdots & \vdots \\ \beta_{N1} & \beta_{N2} & \dots & \beta_{NN} \end{bmatrix}$$

Suppose that (A_j, B) is not controllable. Then by PBH eigenvector test, there exists a non-zero eigenvector ξ of A_j such that $\xi B = 0$. The state matrix of the networked system Ω is given by

$$\Omega = \begin{bmatrix} A_1 + \beta_{11}H & \beta_{12}H & \dots & \dots & \dots & \beta_{1N}H \\ \beta_{21} & A_2 + \beta_{22}H & \dots & \dots & \dots & \beta_{2N}H \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & A_j & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \beta_{N1}H & \beta_{N2}H & \dots & \dots & \dots & A_N + \beta_{NN}H \end{bmatrix}$$

and hence $e_j \otimes \xi$ is a left eigenvector of Ω. Since $\xi B = 0$,

$$(e_j \otimes \xi)(D \otimes B) = e_j D \otimes \xi B = 0$$

Thus by the PBH eigenvector test the networked system is not controllable. □

Example 4.6. (Ajayakumar and George, 2023b) Consider a heterogeneous network system composed of 2 nodes. The state matrices of each node (A_1, A_2), control matrix B and inner coupling matrix H are given by;

$$A_1 = \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } H = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \tag{4.9}$$

The network topology matrix and the external input channel matrix are given by;

$$L = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

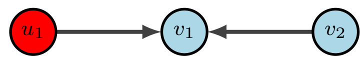

Figure 4.5: The networked system is not controllable with parameters given in (4.9). Observe that there are no edge to node v_2 from node v_1.

There is no edge to node v_2 from node v_1. Also, (A_2, B) is not controllable as the controllability matrix

$$\mathcal{Q}(A_2, B) = [B \mid A_2B] = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

has rank 2. Hence by Theorem 4.6 the networked system is not controllable.

We can verify this by using Kalman's rank condition. The given system can be written in the compact form (4.3), where

$$\Omega = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 3 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix} \text{ and } \Psi = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Then, the controllability matrix,

$$\mathcal{Q}(\Omega, \Psi) = [\Omega \mid \Omega\Psi \mid \Omega^2\Psi \mid \Omega^3\Psi] = \begin{bmatrix} 1 & 2 & 7 & 20 \\ 0 & 3 & 6 & 21 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

has rank 2 and hence the given networked system is not controllable.

We have seen that the controllability of an individual node is necessary when there are no incoming edges to that node. But this is not the case when there are no outgoing edges from a node. Example 4.7 shows that controllability of an individual node is not necessary for network controllability, even if there are no outgoing edges from that node.

Remark 4.4. (Ajayakumar and George, 2023b) If there exists some node v_j with no edge to other nodes, the controllability of (A_j, B) is not necessary for the controllability of the networked system.

Example 4.7. (Ajayakumar and George, 2023b) Consider a heterogeneous network system composed of 2 nodes. The state matrices of each node (A_1, A_2), control matrix B and inner coupling matrix H are given by;

$$A_1 = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{4.10}$$

The network topology matrix and the external input channel matrix are given by;

$$L = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Figure 4.6: The networked system is controllable with parameters given in (4.10). Observe that there are no edge from node v_2 to node v_1.

There is no edge from node v_2 to v_1, and (A_2, B) is not controllable as the controllability matrix

$$\mathcal{Q}(A_2, B) = [B \mid A_2B] = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

has rank 1. The given system can be written in the compact form (4.3), where

$$\Omega = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \text{ and } \Psi = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Then, the controllability matrix,

$$\mathcal{Q}(\Omega, \Psi) = [\Omega \mid \Omega\Psi \mid \Omega^2\Psi \mid \Omega^3\Psi] = \begin{bmatrix} 1 & 0 & 1 & 0 & 3 & 0 & 11 & 0 \\ 0 & 0 & 1 & 0 & 4 & 0 & 15 & 0 \\ 0 & 1 & 0 & 1 & 2 & 1 & 7 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 3 & 0 \end{bmatrix}$$

has rank 4 and hence the given networked system is controllable.

The following theorem addresses a situation where there is an individual node with no outgoing edges. Here controllability of the mentioned individual node becomes a necessary condition for the controllability of the networked system.

Theorem 4.7. *(Ajayakumar and George, 2023b) Suppose that there exists a node v_j from which there is no edge to any other nodes. If $\xi H = 0$ for all left eigenvectors ξ of A_j, then the controllability of (A_j, B) is necessary for the controllability of the networked system.*

Proof. If there exists some node v_j from which no edge to any other nodes, then the network topology matrix L takes the form,

$$L = \begin{bmatrix} \beta_{11} & \beta_{12} & \dots & \beta_{1(j-1)} & 0 & \beta_{1(j+1)} & \dots & \beta_{1N} \\ \beta_{21} & \beta_{22} & \dots & \beta_{2(j-1)} & 0 & \beta_{2(j+1)} & \dots & \beta_{2N} \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ \beta_{N1} & \beta_{N2} & \dots & \beta_{N(j-1)} & 0 & \beta_{N(j+1)} & \dots & \beta_{NN} \end{bmatrix}$$

The state matrix of the networked system Ω is given by,

$$\Omega = \begin{bmatrix} A_1 + \beta_{11}H & \beta_{12}H & \dots & 0 & \dots & \beta_{1N}H \\ \beta_{21}H & A_2 + \beta_{22}H & \dots & 0 & \dots & \beta_{2N}H \\ \vdots & \vdots & \ddots & \vdots & \dots & \vdots \\ \beta_{j1}H & \beta_{j2}H & \dots & A_j & \dots & \beta_{jN}H \\ \vdots & \vdots & \dots & \vdots & \ddots & \vdots \\ \beta_{N1}H & \beta_{N2}H & \dots & 0 & \dots & A_N + \beta_{NN}H \end{bmatrix}$$

Suppose that (A_j, B) is not controllable. Then by PBH eigenvector test there exists a left eigenvector ξ of A_j such that $\xi B = 0$. From the hypothesis of the theorem, $\xi H = 0$ for all left eigenvectors of A_j, $e_j \otimes \xi$ is a left eigenvector of Ω. Also,

$$(e_j \otimes \xi)(D \otimes B) = e_j D \otimes \xi B = 0$$

Hence by PBH eigenvector test the networked system is not controllable.

Example 4.8. (Ajayakumar and George, 2023b) Consider a heterogeneous network system composed of 2 nodes. The state matrices of each node (A_1, A_2), control matrix B and inner coupling matrix H are given by;

$$A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \text{ and } H = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \tag{4.11}$$

The network topology matrix and the external input channel matrix are given by;

$$L = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

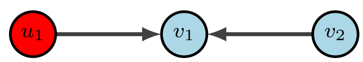

Figure 4.7: The networked system is controllable with parameters given in (4.11). Observe that there are no edge from node v_2 to node v_1.

There is no edge from node v_2 to v_1. Observe that the only left eigenvector of A_2 is $\xi = \begin{bmatrix} 0 & 1 \end{bmatrix}$ and

$$\xi H = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

(A_2, B) is not controllable as the controllability matrix

$$\mathcal{Q}(A_2, B) = [B \mid A_2 B] = \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix}$$

has rank 1. Then by Theorem 4.7, the given networked system is not controllable.

We can verify this by using Kalman's rank condition. The given system can be written in the compact form (4.3), where

$$\Omega = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{ and } \Psi = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Then, the controllability matrix,

$$\mathcal{Q}(\Omega, \Psi) = [\Omega \mid \Omega\Psi \mid \Omega^2\Psi \mid \Omega^3\Psi] = \begin{bmatrix} -1 & -1 & -2 & -3 \\ 0 & -1 & -1 & -1 \\ 0 & -1 & -4 & -8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

has rank 3 and hence the given networked system is not controllable.

□

4.6 Conclusions

Wang et al.(Wang et al., 2016b) derived a necessary and sufficient condition for the controllability of homogeneous networked systems which was later extended to the class of heterogeneous networked systems by Xiang et al.(Xiang et al., 2019b) However, this result was computationally demanding and it had a resemblance with the PBH eigenvector test. Also, the results do not provide much information regarding the effect of network topology, nodal dynamics etc. on the controllability of networked systems. Hao et al.(Hao et al., 2018) proposed a set of conditions that are necessary and sufficient for the controllability of homogeneous networked systems which provided some information regarding the effect of these factors. The obtained results were comparatively easy to verify.

In this chapter, a set of conditions that are necessary and sufficient for the controllability of a class of heterogeneous networked systems where the state matrices can be distinct in each node and the control matrices are identical, is derived. Furthermore, our result extends the study scope to a broader class of systems from the available literature by generalizing the work of Hao et al.(Hao et al., 2018) on controllability of homogeneous LTI networked systems to heterogeneous systems. In addition, controllability results over specific network topologies have been derived for a general heterogeneous networked system. Our result is easy to verify compared to previous findings in the literature. It provides more information about how subsystem dynamics, network topology, and other factors affect the controllability of a networked system. Also, our result can be used to modify a networked system by identifying the nodes that require a control input to make an uncontrollable system into a controllable system, as shown in Example 4.4. Even though our result applies to a broader class of systems, there are some limitations, as mentioned in Remark 4.2, which we hope to rectify in further research. The theoretical results are illustrated using numerical examples.

Chapter 5

Controllability of Heterogeneous Networked Systems with Non-Identical Control Input Matrices

5.1 Introduction

The controllability of networked systems with individual nodes possessing identical control matrices and non-identical state matrices was investigated in the preceding chapter. Having the same control matrix in each individual node was considered as a constraint on the previous results. Nonetheless, the controllability of heterogeneous networked systems — where each node has a different state and control matrix — is studied in this chapter. We use the same methodology as in the previous chapter. First, we will compute the eigenvalues and eigenvectors of the state matrix Ω and then we will use PBH eigenvector test to obtain necessary and sufficient conditions for the controllability of such systems. The obtained results generalizes the results obtained by Hao et al.(Hao et al., 2019) and Ajayakumar et al.(Ajayakumar and George, 2023b). The same methodology is used to obtain simplified controllability conditions for networked systems having an upper/lower triangular matrix as its network topology matrix L.

The heterogeneous networked system models under consideration is formulated in Section 5.2. Controllability of heterogeneous networked systems with non-identical control matrices is discussed in and the controllability results for such systems having non-identical inner coupling matrices and triangular network topology matrix are obtained in Section 5.4. Derived results are substantiated with numerical examples. Conclusions are given in Section 5.5.

5.2 Problem formulation

Consider a heterogeneous networked linear time-invariant system with N nodes, where the i^{th} node is described by the following differential equation:

$$\dot{x}_i(t) = A_i x_i(t) + \sum_{j=1}^{N} \beta_{ij} H x_j(t) + d_i B_i u_i(t), \quad i = 1, 2, \ldots, N \tag{5.1}$$

where, $x_i(t) \in \mathbb{R}^n$ is the state vector and $u_i(t) \in \mathbb{R}^m$ is the external control vector. A_i is an $n \times n$ matrix and B_i is an $n \times m$ matrix called the state matrix and the control matrix of node i respectively.

$$d_i = \begin{cases} 1, & \text{if node i is under control} \\ 0, & \text{otherwise} \end{cases}$$

The connection strength between the nodes j and i is given by $\beta_{ij} \in \mathbb{R}$. If there is a communication from node j to node i, $\beta_{ij} \neq 0$ and otherwise, $\beta_{ij} = 0$, $i, j = 1, 2, \ldots, N$. The $n \times n$ matrix H denotes the inner coupling matrix describing the interconnections among the states $x_j, j = 1, 2, \ldots, N$ of the nodes.

The network topology and external input channels of the networked system (5.1), are given by the $N \times N$ matrices

$$L = [\beta_{ij}] \quad \text{and} \quad D = diag\{d_1, d_2, \ldots, d_N\} \tag{5.2}$$

respectively. If we denote the network state matrix and the total external control input of the networked system (5.1) by $X = \left[x_1^T, \ldots, x_N^T\right]^T$ and $U = \left[u_1^T, \ldots, u_N^T\right]^T$, respectively, using the Kronecker product notation, the system (5.1) can be reduced into the following compact form:

$$\dot{X}(t) = \Omega X(t) + \Psi U(t) \tag{5.3}$$

with,

$$\begin{aligned} \Omega &= \mathcal{A} + L \otimes H \\ \Psi &= (D \otimes I)\mathcal{B} \end{aligned} \tag{5.4}$$

where $\mathcal{A} = blockdiag\{A_1, A_2, \ldots, A_N\}$ and $\mathcal{B} = blockdiag\{B_1, B_2, \ldots, B_N\}$. If the inner coupling matrix is also different in each node, i.e., if the dynamics of the i^{th} node is given by

$$\dot{x}_i(t) = A_i x_i(t) + \sum_{j=1}^{N} \beta_{ij} H_i x_j(t) + d_i B_i u_i(t), \quad i = 1, 2, \ldots, N \tag{5.5}$$

The networked system can be reduced to the compact form (5.3), where

$$\Omega = \begin{bmatrix} A_1 + \beta_{11}H_1 & \beta_{12}H_1 & \ldots & \beta_{1N}H_1 \\ \beta_{21}H_2 & A_2 + \beta_{22}H_2 & \ldots & \beta_{2N}H_2 \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{N1}H_N & \beta_{N2}H_N & \ldots & A_N + \beta_{NN}H_N \end{bmatrix}$$

and $\Psi = (D \otimes I)\mathcal{B}$ with $\mathcal{B} = blockdiag\{B_1, B_2, \ldots, B_N\}$.

5.3 Controllability of Heterogeneous Networked Systems with Non-Identical Control Matrices

In the previous chapter, we studied the controllability of (5.3) when the network topology is triangularizable, and the system parameter matrices satisfy certain conditions. There the control input matrices were assumed to be identical in each node. Here, we will extend this result to a system where each node has different control matrix.

Suppose that the network topology matrix L is triangularizable. That is, there exists a non-singular matrix T such that $TLT^{-1} = J$, where $J = uppertriang\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$ is the Jordan Canonical Form of L. Let $\sigma(A_i + \lambda_i H) = \{\mu_i^1, \ldots, \mu_i^{q_i}\}$ denotes the set of eigenvalues of $A_i + \lambda_i H, i = 1, 2, \ldots, N$ and $\xi_{ij}^k, k = 1, \ldots, \gamma_{ij}$ be the left eigenvectors of $A_i + \lambda_i H$ corresponding to $\mu_i^j, j = 1, \ldots, q_i, i = 1, \ldots, N$, where $\gamma_{ij} \geq 1$ is the geometric multiplicity of the eigenvalue μ_i^j. With the aid of Theorem 4.3, we can prove the following necessary and sufficient conditions for controllability of the networked system (5.3).

Theorem 5.1. *(Ajayakumar and George, 2023a) Let T be a non-singular matrix triagularizing matrix L such that $T \otimes I$ commutes with $\mathcal{A}$. If J contains a Jordan block of order $l \geq 2$ corresponding to the eigenvalue λ_{i_0} of L, then assume that $\xi_{ij}^k H = 0$ for all $i = i_0, i_0 + 1, \ldots, i_0 + l - 1, j = 1, 2, \ldots, q_i, k = 1, 2, \ldots, \gamma_{ij}$, where $\xi_{ij}^k, i = 1, 2, \ldots, N, j = 1, 2, \ldots, q_i, k = 1, 2, \ldots, \gamma_{ij}$ are the left eigenvectors of $A_i + \lambda_i H$ corresponding to the eigenvalues $\mu_i^j, i = 1, 2, \ldots, N, j = 1, 2, \ldots, q_i$. Then the networked system* (5.3) *is controllable if and only if* *(i)* $e_iTD \neq 0$ *for all* $i = 1, \ldots, N$

(ii) *For a fixed* i*, each left eigenvector* ξ *of* $A_i + \lambda_i H$*,* $\xi Bj \neq 0$ *for some* $j \in \{1, 2, \cdots, N\}$ *with* $[e_iTD]_j \neq 0$ *; and*

(iii) *If matrices* $A_{i_1} + \lambda_{i_1} H, A_{i_2} + \lambda_{i_2} H, \ldots, A_{i_p} + \lambda_{i_p} H(\lambda_{i_k} \in \sigma(C), k = 1, \ldots, p,\ where\, p > 1)$ *have a common eigenvalue* ρ*, then* $(e_{i_1}TD \otimes \xi^1_{i_1})\mathcal{B}, \cdots, (e_{i_1}TD \otimes \xi^{\gamma_{i_1}}_{i_1})\mathcal{B}, \ldots, (e_{i_p}TD \otimes \xi^1_{i_p})\mathcal{B}, \ldots, (e_{i_p}TD \otimes \xi^{\gamma_{i_p}}_{i_p})\mathcal{B}$ *are linearly independent vectors, where* $\gamma_{i_k} \geq 1$ *is the geometric multiplicity of the eigenvalue* ρ *for the matrix* $A_{i_k} + \lambda_{i_k} H$ *and* $\xi^l_{i_k}(l = 1, \ldots, \gamma_{i_k})$ *are the left eigenvectors of* $A_{i_k} + \lambda_{i_k} H$ *corresponding to* $\rho, k = 1, \ldots, p$*.*

Proof. **(Necessary part)** Fix i. Let ξ be an arbitrary left eigenvector of $A_i + \lambda_i H$. From Theorem 4.3, we have that that $e_iT \otimes \xi$ is a left eigenvector of Ω. By PBH eigenvector test, for the networked system (5.3) to be controllable, we must have

$$(e_iT \otimes \xi)(D \otimes I)\mathcal{B} = (e_iTD \otimes \xi)\mathcal{B} \neq 0$$

This implies that $e_iTD \neq 0$ and $\xi B_j \neq 0$ for some $j \in \{1, 2, \ldots, N\}$ with $[e_iTD]_j \neq 0$.

Now, suppose that the matrices $A_{i_1} + \lambda_{i_1} H, \ldots, A_{i_p} + \lambda_{i_p} H(\lambda_{i_k} \in \sigma(L), k = 1, \ldots, p,$ where $p > 1)$ have a common eigenvalue ρ. Then the left eigenvectors of Ω corresponding to ρ can be expressed as a linear combination in the form $\sum_{k=1}^{p} \sum_{l=1}^{\gamma_{i_k}} \alpha_{kl}(e_{i_k}T \otimes \xi^l_{i_k})$, where $\alpha_{kl} \in \mathbb{R}(k = 1, \ldots, p, l = 1, \ldots, \gamma_{i_k})$ are scalars, not all are zero and $\xi^1_{i_k}, \ldots, \xi^{\gamma_{i_k}}_{i_k}$, are the eigenvectors of $A_{i_k} + \lambda_{i_k} H$ corresponding to the eigenvalue ρ, where $k = 1, \ldots, p$. If the networked system is controllable, then

$$\left[\sum_{k=1}^{p} \sum_{l=1}^{\gamma_{i_k}} \alpha_{kl}(e_{i_k}T \otimes \xi^l_{i_k})\right](D \otimes I)\mathcal{B} \neq 0$$

Consequently, we have

$$\sum_{k=1}^{p} \sum_{l=1}^{\gamma_{ik}} \alpha_{kl} \left\{ \left[(e_{i_k}TD) \otimes (\xi^l_{ik}) \right] \mathcal{B} \right\} \neq 0$$

for any scalars $\alpha_{kl} \in \mathbb{R}(k = 1, \ldots, p, l = 1, \ldots, \gamma_{ik})$, not all of them are zero. This implies that the vectors $[(e_{i1}TD) \otimes (\xi^1_{i1})]\mathcal{B}, \ldots, [(e_{i1}TD) \otimes (\xi^{\gamma_{i1}}_{i1})]\mathcal{B}, \ldots, \left[(e_{ip}TD) \otimes (\xi^1_{ip})\right]\mathcal{B}, \ldots, \left[(e_{ip}TD) \otimes (\xi^{\gamma_{ip}}_{ip})\right]\mathcal{B}$ are linearly independent in $\mathbb{R}^{Nn}$.

(Sufficiency part) To prove the converse part, we will show that if the networked system is uncontrollable, at least one condition in Theorem 5.1 does not hold. Suppose that the networked system (5.3) is not controllable. Then by PBH eigenvector test, there exists a left eigenpair $(\tilde{\mu}, \tilde{v})$ of Ω, such that $\tilde{v}\Psi = 0$.

If $\tilde{\mu} \in \sigma(A_{i_0} + \lambda_{i_0} H)$ and $\tilde{\mu} \notin \sigma(A_1 + \lambda_1 H) \cup \ldots \cup \sigma(A_{i_0-1} + \lambda_{i_0-1} H) \cup \sigma(A_{i_0+1} + \lambda_{i_0+1} H) \cup \ldots \cup \sigma(A_{i_N} + \lambda_{i_N} H)$. Again $\tilde{v}$ can be written as a linear combination, $\tilde{v} = \sum_{l=1}^{\gamma_{i_0 j_0}} \alpha_0^l (e_{i_0} T \otimes \xi_{i_0 j_0}^l)$, where $\xi_{i_0 j_0}^1, \ldots, \xi_{i_0 j_0}^{\gamma_{i_0 j_0}}$ of left eigenvectors of $A_{i_0} + \lambda_{i_0} H$ corresponding to $\tilde{\mu}$, where, $\left[\alpha_0^1, \ldots, \alpha_0^{\gamma_{i_0 j_0}}\right]$ is some non-zero vector. Now $\tilde{v}\Psi = 0$ implies

$$\sum_{l=1}^{\gamma_{i_0 j_0}} \alpha_0^l (e_{i_0} T \otimes \xi_{i_0 j_0}^l)(D \otimes I)B = \left[(e_{i_0} T D) \otimes \left(\sum_{l=1}^{\gamma_{i_0 j_0}} \alpha_0^l \xi_{i_0 j_0}^l\right)\right] \mathcal{B} = 0$$

This implies that either $e_{i_0} T D = 0$ or $\left(\sum_{l=1}^{\gamma_{i_0 j_0}} \alpha_0^l \xi_{i_0 j_0}^l\right) B_j = 0$ for all $j \in \{1, 2, \cdots, N\}$ with $[e_i T D]_j \neq 0$. Keep in mind that $\sum_{l=1}^{\gamma_{i_0 j_0}} \alpha_0^l \xi_{i_0 j_0}^l$ is a left eigenvector of $A_{i_0} + \lambda_{i_0} H$. Thus, if the networked system is uncontrollable, then either condition (i) or condition (ii) does not hold true.

Let $\tilde{\mu}$ be the common eigenvalue of the matrices $A_{i_1} + \lambda_{i_1} H, \ldots, A_{i_p} + \lambda_{i_p} H (\lambda_{i_k} \in \sigma(L)$, for $k = 1, \ldots, p, p > 1)$. Also, let the eigenvectors of $A_{i_k} + \lambda_{i_k}$ corresponding to $\tilde{\mu}$ are $\xi_{i_k}^1, \ldots, \xi_{i_k}^{\gamma_{i_k}}$, where $k = 1, \ldots, p$. Since $\tilde{v}$ can be expressed in the form

$$\tilde{v} = \sum_{k=1}^{p} \sum_{l=1}^{\gamma_{i_k}} \alpha_0^{kl} \left(e_{i_k} T \otimes \xi_{ik}^l\right)$$

where $\alpha_0^{kl} (l = 1, \ldots, \gamma_{i_k}, k = 1, \ldots, p)$ are some scalars, which are not all zero. Then $\tilde{v}\Psi = 0$ implies that there exists a non-zero vector $\left[\alpha_0^{11}, \ldots, \alpha_0^{1\gamma_{i_1}}, \ldots, \alpha_0^{p1}, \ldots, \alpha_0^{p\gamma_{i_p}}\right]$ such that

$$\left[\sum_{k=1}^{p} \sum_{l=1}^{\gamma_{i_k}} \alpha_0^{kl} \left(e_{i_k} T \otimes \xi_{i_k}^l\right)\right] (D \otimes I)\mathcal{B} = \sum_{k=1}^{p} \sum_{l=1}^{\gamma_{i_k}} \alpha_0^{kl} \left\{\left[(e_{i_k} T D) \otimes (\xi_{i_k}^l)\right] \mathcal{B}\right\} = 0$$

This implies that $(e_{i_1} T D \otimes \xi_{i_1}^1)\mathcal{B}, \ldots, (e_{i_1} T D \otimes \xi_{i_1}^{\gamma_{i_1}})\mathcal{B}, \ldots, (e_{i_p} T D \otimes \xi_{i_p}^1)\mathcal{B}, \ldots, (e_{i_p} T D \otimes \xi_{i_p}^{\gamma_{i_p}})\mathcal{B}$ are linearly dependent. Thus at least one condition in Theorem 5.1 does not hold true, when the networked system is not controllable. □

The following numerical examples illustrate the controllability result obtained in Theorem 5.1.

Example 5.1. (Ajayakumar and George, 2023a) Consider a heterogeneous networked system with 3 nodes with the following dynamics; The state matrices and control matrices of each nodes are given by,

$$A_1 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \ A_2 = A_3 = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \ B_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \ B_2 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \ B_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad (5.6)$$

The network topology matrix, inner-coupling matrix and the external control input matrix are respectively,

$$L = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \ H = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (5.7)$$

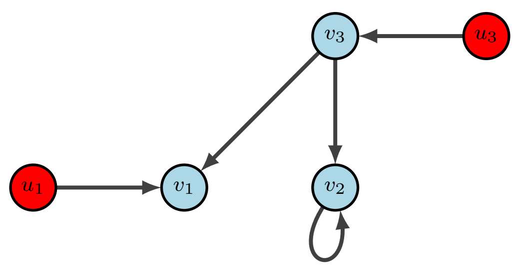

Figure 5.1: Controllable heterogeneous networked system with triangularizable network topology L and node dynamics as in (5.6)-(5.7).

There exists a non-singular matrix $T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ such that $TLT^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. We have, $\lambda_1 = 0, \lambda_2 = 0$ and $\lambda_3 = 1$. Clearly, J contains a Jordan block of order 2 corresponding to 0. $\xi_{11}^1 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$ is the only left eigenvector of the matrix $A_1 + \lambda_1 H = A_1$ and $\xi_{11}^1 H = 0$. Also $T \otimes I$ commutes with $\mathcal{A}$. Then,

(i) as $TD = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $e_i TD \neq 0$ for all $i = 1, 2, 3$.

(ii) for $A_1 + \lambda_1 H = A_1 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, the only left eigenvector is $\xi_{11}^1 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$. We have $[e_1 TD]_1 \neq 0$ and $\xi_{11}^1 B_1 \neq 0$.

For the matrix $A_2 + \lambda_2 H = A_2 = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ the left eigenvectors are respectively

$$\xi_{21}^1 = \begin{bmatrix} 0.44062 & 0.828911 & 1 \end{bmatrix}$$
$$\xi_{22}^1 = \begin{bmatrix} -0.72031 - 0.784805i & -0.914456 + 1.47641i & 1 \end{bmatrix}$$

and

$$\xi_{23}^1 = \begin{bmatrix} -0.72031 + 0.784805i & -0.914456 - 1.47641i & 1 \end{bmatrix}$$

We have $[e_2 TD]_3 \neq 0$ and $\xi_{21}^1 B_3, \xi_{22}^1 B_3, \xi_{23}^1 B_3 \neq 0$.

For the matrix $A_3 + \lambda_3 H = A_3 + H = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 2 \\ 1 & 1 & 2 \end{bmatrix}$, the left eigenvectors are respectively

$$\xi_{31}^1 = \begin{bmatrix} 0.720551 & 1.09001 & 1 \end{bmatrix}$$
$$\xi_{32}^1 = \begin{bmatrix} -0.0875483 - 0.34424i & -0.681369 + 0.450503i & 1 \end{bmatrix}$$

and

$$\xi_{33}^1 = \begin{bmatrix} -0.0875483 + 0.34424i & -0.681369 - 0.450503i & 1 \end{bmatrix}$$

We have $[e_3 TD]_3 \neq 0$ and $\xi_{31}^1 B_3, \xi_{32}^1 B_3, \xi_{33}^1 B_3 \neq 0$.

(iii) as the matrices A_1, A_2 and $A_3 + H$ do not have any common eigenvalues, third condition of Theorem 5.1 is automatically satisfied.

Thus all the conditions of Theorem 5.1 are satisfied and hence the given networked system is controllable.

We can also use the Kalaman's rank condition to verify the controllability of the given networked system. The system can be written in the compact form (5.3), where

$$\Omega = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 3 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 \end{bmatrix} \text{ and } \Psi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

We can see that the controllability matrix $\mathcal{Q}(\Omega, \Psi)$ has rank 9 and hence the given networked system is controllable.

The following illustration demonstrates how Theorem 5.1 can be used to make an uncontrollable system controllable.

Example 5.2. (Ajayakumar and George, 2023a) Consider a heterogeneous networked system with 3 nodes with the following dynamics; The state matrices and control matrices of each nodes are given by,

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & -1 \end{bmatrix}, \; A_2 = A_3 = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & 2 & -1 \end{bmatrix}, \; B_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \; B_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \tag{5.8}$$

The network topology matrix, inner-coupling matrix and the external control input matrix are respectively,

$$L = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \; H = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{5.9}$$

There exists $T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$ such that $TLT^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. The eigenvalues of L are, $\lambda_1 = 1, \lambda_2 = 1$ and $\lambda_3 = -1$. Clearly, J contains a Jordan block of order 2.

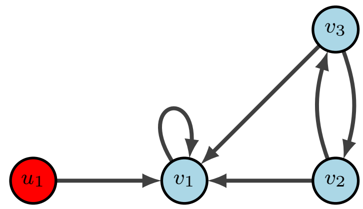

Figure 5.2: Uncontrollable heterogeneous networked system with triangularizable network topology L and node dynamics as in (5.8)-(5.9).

Observe that $\xi_{11}^1 = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}$ is the only left eigenvector corresponding to the matrix $A_1 + H$ and $\xi_{11}^1 H = 0$. Also $T \otimes I$ commutes with $\mathcal{A}$. Here, as $TD = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, we have $e_2 TD = e_3 TD = 0$. Then, by Theorem 5.1, the networked system is not controllable. It is easy to observe that, either node v_2 or v_3 must be supplied with a control input, so that $e_i TD \neq 0$ for all $i = 1, 2, 3$. Suppose that node v_2 is supplied with an external control input matrix. That is, $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Then the network graph is as follows:

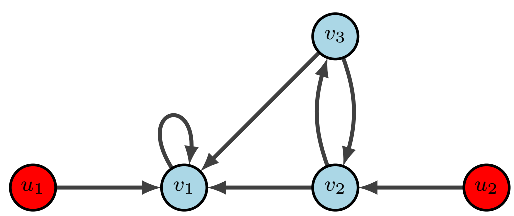

Figure 5.3: Network graph of the given system with control inputs in mode v_1 and v_2.

Even after giving a control input in node v_2 the networked system is not controllable as $[e_1 TD]_1$ is the only non-zero entry in $e_1 TD$ and $\xi_{11}^1 B_1 = 0$. If we could change the control input matrix B_1 so that $\xi_{11}^1 B_1 \neq 0$, we can make this uncontrollable system to a controllable system. For example, consider $B_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. Then,

(i) as $TD = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & 0 \end{bmatrix}$, $e_i TD \neq 0$ for all $i = 1, 2, 3$.

(ii) for $A_1 + H = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & -1 \\ 0 & 1 & 0 \end{bmatrix}$, the only left eigenvector is $\xi_{11}^1 = \begin{bmatrix} 0 & -1 & 1 \end{bmatrix}$. We have $[e_1 TD]_1 \neq 0$ and $\xi_{11}^1 B_1 \neq 0$.

For $A_2 + H = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 0 \\ 0 & 3 & 0 \end{bmatrix}$, the left eigenvectors are

$$\xi_{21}^1 = \begin{bmatrix} 3.90547 & 5.67363 & 1 \end{bmatrix}$$
$$\xi_{22}^1 = \begin{bmatrix} -0.452737 + 1.15383i & -0.336813 - 1.0993i & 1 \end{bmatrix}$$

and

$$\xi_{23}^1 = \begin{bmatrix} -0.452737 - 1.15383i & -0.336813 + 1.0993i & 1 \end{bmatrix}$$

We have $[e_2 TD]_2 \neq 0$ and $\xi_{21}^1 B_2, \xi_{22}^1 B_2, \xi_{23}^1 B_2 \neq 0$.

and for the matrix $A_3 - H = \begin{bmatrix} -1 & 0 & 1 \\ 2 & 0 & -2 \\ 0 & 1 & -2 \end{bmatrix}$, the left eigenvectors are

$$\xi_{31}^1 = \begin{bmatrix} 2 & 1 & 0 \end{bmatrix}$$
$$\xi_{32}^1 = \begin{bmatrix} -0.25 + 0.661438i & -0.375 - 0.330719i & 1 \end{bmatrix}$$

and

$$\xi_{33}^1 = \begin{bmatrix} -0.25 - 0.661438i & -0.375 + 0.330719i & 1 \end{bmatrix}$$

We have $[e_3 TD]_2 \neq 0$ and $\xi_{31}^1 B_2, \xi_{32}^1 B_2, \xi_{33}^1 B_2 \neq 0$.

(iii) as the matrices $A_1 + H, A_2 + H$ and $A_3 - H$ do not have any common eigenvalues, third condition of Theorem 5.1 is satisfied.

Thus all the conditions of Theorem 5.1 are satisfied and hence the system is controllable.

Thus, with the help of conditions in Theorem 5.1 we can modify the system components in order to make an uncontrollable system controllable. Now, suppose that $(A_i + \lambda_i H, B_j)$ is controllable for some $j \in \{1, 2, \cdots, N\}$ with $[e_i TD]_j \neq 0$. Then by PBH eigenvector test, for each left eigenvector ξ of $A_i + \lambda_i H$, $\xi Bj \neq 0$. From this idea, we can derive the following result as a corollary of Theorem 5.1, which gives a sufficient condition for controllability.

Corollary 5.1. *(Ajayakumar and George, 2023a) Let T be a non-singular matrix triagularizing matrix L such that $T \otimes I$ commutes with $\mathcal{A}$. If J contains a Jordan block of order $l \geq 2$ corresponding to the eigenvalue λ_{i_0} of L, then assume that $\xi_{ij}^k H = 0$ for all $i = i_0, i_0+1, \ldots, i_0+l-1, j = 1,2,\ldots,q_i, k = 1,2,\ldots,\gamma_{ij}$, where $\xi_{ij}^k, i = 1,2,\ldots,N, j = 1,2,\ldots,q_i, k = 1,2,\ldots,\gamma_{ij}$ are the left eigenvectors of $A_i + \lambda_i H$ corresponding to the eigenvalues $\mu_i^j, i = 1,2,\ldots,N, j = 1,2,\ldots,q_i$. Then the networked system* (5.3) *is controllable if the following conditions are satisfied.*

(i) $e_i TD \neq 0$ *for all* $i = 1,\ldots,N$

(ii) For a fixed i, $(A_i + \lambda_i H, B_j)$ is controllable for some $j \in \{1,2,\cdots,N\}$ with $[e_i TD]_j \neq 0$; and

(iii) If matrices $A_{i_1}+\lambda_{i_1}H, A_{i_2}+\lambda_{i_2}H, \ldots, A_{i_p}+\lambda_{i_p}H(\lambda_{i_k} \in \sigma(L), k = 1,\ldots,p,$ where $p > 1)$ have a common eigenvalue ρ, then $(e_{i_1}TD \otimes \xi_{i_1}^1)\mathcal{B}, \cdots, (e_{i_1}TD \otimes \xi_{i_1}^{\gamma_{i_1}})\mathcal{B}, \ldots, (e_{i_p}TD \otimes \xi_{i_p}^1)\mathcal{B}, \ldots, (e_{i_p}TD \otimes \xi_{i_p}^{\gamma_{i_p}})\mathcal{B}$ are linearly independent vectors, where $\gamma_{i_k} \geq 1$ is the geometric multiplicity of the eigenvalue ρ for the matrix $A_{i_k} + \lambda_{i_k}H$ and $\xi_{i_k}^l (l = 1,\ldots,\gamma_{i_k})$ are the left eigenvectors of $A_{i_k} + \lambda_{i_k}H$ corresponding to $\rho, k = 1,\ldots,p$.

Example 5.3. (Ajayakumar and George, 2023a) Consider a networked system with 3 nodes, where the dynamics of the system is given as follows;

$$A_1 = A_2 = A_3 = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}, \ B_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \ B_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ B_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \tag{5.10}$$

The network topology matrix, inner-coupling matrix and the external input matrix are given by ,

$$L = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \ H = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{5.11}$$

L is diagonalizable with $T = \begin{bmatrix} 0 & 1 & -1 \\ \frac{\sqrt{3}}{2} & 0 & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & 0 & \frac{\sqrt{3}}{2} \end{bmatrix}$, such that $TLT^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = J$.
We have $\lambda_1 = 0$, $\lambda_2 = 1$ and $\lambda_1 = -1$. Clearly, J does not contain any Jordan blocks and $T \otimes I$ commutes with $\mathcal{A}$. Then,

(i) as $TD = \begin{bmatrix} 0 & 1 & 0 \\ \frac{\sqrt{3}}{2} & 0 & 0 \\ -\frac{\sqrt{3}}{2} & 0 & 0 \end{bmatrix}$, $e_iTD \neq 0$ for all $i = 1, 2, 3$.

(ii) We have $[e_1TD]_2, [e_2TD]_1, [e_3TD]_1 \neq 0$. Here $(A_1, B_2), (A_2 + H, B_1)$ and $(A_1 - H, B_2)$ are controllable.

(iii) Here A_1 and $A_3 - H$ has a common eigenvalue, $\rho = 1$. The corresponding left eigenvectors are respectively, $\xi = \begin{bmatrix} -2 & 0 & 1 \end{bmatrix}$ and $\nu = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$. Clearly, $(e_1TD \otimes \xi)\mathcal{B} \neq 0$ and $(e_3TD \otimes \nu)\mathcal{B} \neq 0$.

Thus all the conditions of Corollary 5.1 are satisfied and hence the given system is controllable.

We can also use the Kalaman's rank condition to verify the controllability of the given networked system. The system can be written in the compact form (5.3), where

$$\Omega = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 2 & 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 2 & 1 & 2 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 2 & 1 \end{bmatrix} \text{ and } \Psi = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We can see that the controllability matrix $\mathcal{Q}(\Omega, \Psi)$ has rank 9 and hence the given networked system is controllable.

If $B_i = B$ for all $i = 1, 2, \ldots, N$, then the Theorem 4.4 can be obtained as a consequence of Corollary 5.1.

Theorem 5.2. *(Ajayakumar and George, 2023b) Let T be a non-singular matrix triagularizing matrix L such that $T \otimes I$ commutes with $\mathcal{A}$. If J contains a Jordan block of order $l \geq 2$ corresponding to the eigenvalue λ_{i_0} of L, then assume that $\xi_{ij}^k H = 0$ for all $i = i_0, i_0 + 1, \ldots, i_0 + l - 1, j = 1, 2, \ldots, q_i, k = 1, 2, \ldots, \gamma_{ij}$, where $\xi_{ij}^k, i = 1, 2, \ldots, N, j = 1, 2, \ldots, q_i, k = 1, 2, \ldots, \gamma_{ij}$ are the left eigenvectors of $A_i + \lambda_i H$ corresponding to the eigenvalues $\mu_i^j, i = 1, 2, \ldots, N, j = 1, 2, \ldots, q_i$. Then the networked system* (5.3) *is controllable if and only if* *(i)* $e_i TD \neq 0$ *for all* $i = 1, \ldots, N$

(ii) $(A_i + \lambda_i H, B)$ *is controllable, for* $i = 1, 2, \cdots, N$*; and*

(iii) If matrices $A_{i_1}+\lambda_{i_1}H, A_{i_2}+\lambda_{i_2}H, \ldots, A_{i_p}+\lambda_{i_p}H(\lambda_{i_k} \in \sigma(L), k = 1, \ldots, p,\ where\, p > 1)$ *have a common eigenvalue* ρ*, then* $(e_{i_1}TD)\otimes(\xi_{i_1}^1 B), \ldots, (e_{i_1}TD)\otimes(\xi_{i_1}^{\gamma_{i_1}} B), \ldots, (e_{i_p}TD)\otimes(\xi_{i_p}^1 B), \ldots, (e_{i_p}TD) \otimes (\xi_{i_p}^{\gamma_{i_p}} B)$ *are linearly independent vectors, where* $\gamma_{i_k} \geq 1$ *is the geometric multiplicity of the eigenvalue* ρ *for the matrix* $A_{i_k} + \lambda_{i_k} H$ *and* $\xi_{i_k}^l (l = 1, \ldots, \gamma_{i_k})$ *are the left eigenvectors of* $A_{i_k} + \lambda_{i_k} H$ *corresponding to* $\rho, k = 1, \ldots, p$*.*

5.4 Controllability of Heterogeneous Networked Systems with Triangular Network Topology

Now we will discuss the controllability of system (5.5), when the network topology is given by an upper/lower triangular matrix and the state matrices have certain properties. Here also, we will characterize the controllability of networked system interms of eigenvalues and eigenvectors of the state matrix Ω.

Theorem 5.3. *(Ajayakumar and George, 2023a) Assume that* L *is an upper triangular matrix. Let* $\sigma(A_i + \beta_{ii} H_i) = \{\mu_i^1, \ldots, \mu_i^{q_i}\}$ *be the set of eigenvalues of* $A_i + \beta_{ii} H_i, i = 1, 2, \ldots, N$*. Then the set of all eigenvalues of* Ω *is given by*

$$\sigma(\Omega) = \{\mu_1^1, \mu_1^2, \ldots, \mu_1^{q_1}, \ldots, \mu_N^1, \mu_N^2, \ldots, \mu_N^{q_N}\}$$

Let $\xi_{ij}^k, k = 1, 2, \ldots, \gamma_{ij}$ *be the left eigenvectors of* $A_i+\beta_{ii}H_i$ *associated with the eigenvalue* μ_i^j*, where* γ_{ij} *is the geometric multiplicity of the eigenvalue* μ_i^j *for the matrix* $A_i + \beta_{ii} H_i$*. If* $\xi_{ij}^k H_i = 0$*, for* $i = 1, 2, \ldots, N-1, j = 1, 2, \ldots, q_i, k = 1, 2, \ldots, \gamma_{ij}$*, then* $e_i \otimes \xi_{ij}^1, e_i \otimes \xi_{ij}^1, \ldots e_i \otimes \xi_{ij}^{\gamma_{ij}}$*, are the left eigenvectors of* Ω *associated with the eigenvalues* μ_i^j*.*

Proof. Suppose that L is an upper triangular matrix, say $L = \begin{bmatrix} \beta_{11} & \beta_{12} & \ldots & \beta_{1N} \\ 0 & \beta_{22} & \ldots & \beta_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \beta_{NN} \end{bmatrix}$.

Then,

$$\Omega = \begin{bmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & A_N \end{bmatrix} + \begin{bmatrix} \beta_{11}H_1 & \beta_{12}H_1 & \dots & \beta_{1N}H_1 \\ 0 & \beta_{22}H_2 & \dots & \beta_{2N}H_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \beta_{NN}H_N \end{bmatrix}$$
$$= \begin{bmatrix} A_1 + \beta_{11}H_1 & \beta_{12}H_1 & \dots & \beta_{1n}H_1 \\ 0 & A_2 + \beta_{22}H_2 & \dots & \beta_{2n}H_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_N + \beta_{NN}H_N \end{bmatrix}$$

is a block upper triangular matrix. Therefore the eigenvalues of Ω are precisely the eigenvalues of the matrices $A_i+\beta_{ii}H_i, i = 1, 2, \dots, N$. That is, if $\sigma(A_i+\beta_{ii}H_i) = \{\mu_i^1, \mu_i^2, \dots, \mu_i^{q_i}\}$ are the eigenvalues of $A_i + \beta_{ii}H_i, i = 1, 2, \dots, N$, then

$$\sigma(\Omega) = \cup_{i=1}^{N}\sigma(A_i + \beta_{ii}H_i) = \left\{\mu_1^1, \mu_1^2, \dots, \mu_1^{q_1}, \dots, \mu_N^1, \mu_N^2, \dots, \mu_N^{q_N}\right\}$$

are the eigenvalues of Ω. Now, if $\xi_{ij}^k, k = 1, 2, \dots, \gamma_{ij}$ represents the left eigenvectors of $A_i + \beta_{ii}H$ associated with the eigenvalue μ_i^j, then clearly $e_N \otimes \xi_{Nj}^1, e_N \otimes \xi_{Nj}^2, \dots, e_N \otimes \xi_{Nj}^{\gamma_{Nj}}$ are left eigenvectors of Ω corresponding to the eigenvalue μ_N^j. If $\xi_{ij}^k H_i = 0, k = 1, 2, \dots, \gamma_{ij}, i = 1, 2, \dots, N-1, j = 1, 2, \dots, q_i$, then $e_i \otimes \xi_{ij}^1, e_i \otimes \xi_{ij}^2, \dots e_i \otimes \xi_{ij}^{\gamma_{ij}}$ are left eigenvectors of Ω associated with the eigenvalue μ_i^j.

Now, we will prove that the the only linearly independent left eigenvectors of Ω are of the form $e_i \otimes \xi$, where ξ is a left eigenvector of $A_i + \beta_{ii}H_i$ for some i. For, take $\xi_1, \xi_2, \dots, \xi_N \in \mathbb{R}^n$, such that $\xi = \begin{bmatrix} \xi_1 & \xi_2 \dots \xi_N \end{bmatrix} \in \mathbb{R}^{Nn}$ is a left eigenvector of Ω. Then $\xi\Omega = \mu\xi$ for some eigenvalue μ of Ω implies that

$$\begin{bmatrix} \xi_1 \left(A_1 + \beta_{11}H_1\right) \\ \xi_1 H + \xi_2 \left(A_2 + \beta_{22}H_2\right) \\ \vdots \\ \sum_{i=1}^{N-1} \xi_i H + \xi_N \left(A_N + \beta_{NN}H_N\right) \end{bmatrix}^T = \mu \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_N \end{bmatrix}$$

Then, clearly μ is an eigenvalue of $A_1 + \beta_{11}H_1$ with left eigenvector ξ_1. Then by our hypothesis, $\xi_1 H_1 = 0$. Thus

$$\xi_1 H_1 + \xi_2 \left(A_2 + \beta_{22}H_2\right) = \xi_2 \left(A_2 + \beta_{22}H_2\right) = \mu\xi_2$$

implies that μ is an eigenvalue of $A_2+\beta_{22}H_2$ with left eigenvector ξ_2. Proceeding like this, we get μ is an eigenvalue of $A_i+\beta_{ii}H_i$ for all $i=1,2,\dots,N$ with ξ_i as left eigenvector. Then ξ can be expressed as $\xi=\sum_{i=1}^{N}e_i\otimes\xi_i$, and we have already seen that $e_i\otimes\xi_i$ are left eigenvectors of Ω for any left eigenvector ξ_i of $A_i+\beta_{ii}H_i$. Thus if $A_i+\beta_{ii}H_i, i=1,2,\dots,N$ does not have any common eigenvalue, the only left eigenvectors of Ω are $e_i\otimes\xi_{ij}^k, i=1,2,\dots,N, j=1,2,\dots,q_i, k=1,2,\dots,\gamma_{ij}$, where $\xi_{ij}^k, k=1,2,\dots,\gamma_{ij}$ are the linearly independent left eigenvectors of $A_i+\beta_{ii}H$ corresponding to the eigenvalue μ_i^j. Now suppose that $A_{i_1}+\beta_{i_1i_1}H, A_{i_2}+\beta_{i_2i_2}H,\dots,A_{i_r}+\beta_{i_ri_r}H$ have a common eigenvalue μ with left eigenvectors $\xi_{i_1},\xi_{i_2},\dots,\xi_{i_r}$ respectively, where $i_0,i_1,\dots,i_r\in\{1,2,\dots,N\}$. Then $\sum_{\alpha=1}^{r}e_{i_\alpha}\otimes\xi_{i_\alpha}$ is a left eigenvector of Ω corresponding to the eigenvalue μ. □

Theorem 5.4. *(Ajayakumar and George, 2023a) Let L be an upper\lower triangular matrix. Suppose the eigenvectors of $A_i+\beta_{ii}H_i$ satisfy the conditions given in Theorem 5.3, then the networked system* (5.3) *is controllable if and only if*

(i) Every node have external control input.

(ii) $(A_i+\beta_{ii}H_i, B_i)$ is controllable for all $i=1,2,\dots N$.

Proof. Suppose that the networked system (5.3) is controllable and suppose that $d_i=0$ for some i, say i_0, i.e., $d_{i_0}=0$. Then the control matrix for the networked system (5.3) is given by

$$G=\begin{bmatrix} d_1B_1 & 0 & \dots & 0 & \dots & 0\\ 0 & d_2B_2 & \dots & 0 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & \dots & d_{i_0}B_{i_0} & \dots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & 0 & \dots & d_NB_N \end{bmatrix}$$
$$=\begin{bmatrix} d_1B_1 & 0 & \dots & 0 & \dots & 0\\ 0 & d_2B_2 & \dots & 0 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & \dots & 0 & \dots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & 0 & \dots & d_NB_N \end{bmatrix}$$

We have proved that $e_{i_0} \otimes \xi^k_{i_0 j}, k = 1, 2, \ldots, \gamma_{i_0 j}$ are left eigenvectors of Φ corresponding to the eigenvalue $\mu^j_{i_0}$, where $j = 1, 2, \ldots, q_{i_0}$. Observe that for any $j = 1, 2, \ldots, q_{i_0}, k = 1, 2, \ldots, \gamma_{i_0 j}$, $\left(e_{i_0} \otimes \xi^k_{i_0 j}\right) \Psi = 0$. Then by PBH eigenvector test, the given system is not controllable, which is a contradiction.

Now suppose that $(A_i + \beta_{ii} H_i, B_i)$ is not controllable for some i, say i_1. Again by PBH eigenvector test, for some eigenvalue $\mu^{j_1}_{i_1}$ (where $j_1 \in \{1, 2, \ldots, q_{i_1}\}$) of $A_{i_1} + \beta_{i_1 i_1} H_{i_1}$ there exists a left eigenvector $\xi^{k_1}_{i_1 j_1}$ (where $k_1 \in \{1, 2, \ldots, \gamma_{i_1 j_1}\}$) such that $\xi^{k_1}_{i_1 j_1} B_{i_1} = 0$. Then clearly $\left(e_{i_1} \otimes \xi^{k_1}_{i_1 j_1}\right) G = 0$, which is a contradiction. Conversely, suppose that both (i) and (ii) are satisfied. We have the left eigenvectors of Φ are $e_i \otimes \xi^k_{ij}$, where $i = 1, 2, \ldots, N, j = 1, 2, \ldots, q_i, k = 1, 2, \ldots, \gamma_{ij}$ or their linear combinations. Now $\left(e_i \otimes \xi^k_{ij}\right) G = 0$ if and only if either $d_i = 0, \xi^k_{ij} B_i = 0$ or both for some i. Both these situations contradicts our hypothesis. Then by PBH eigenvector test, system (5.3) is controllable. □

Example 5.4. (Ajayakumar and George, 2023a) Consider a heterogeneous networked system with 3 nodes, where the state matrices and control matrices are given by;

$$A_1 = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & 2 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, A_3 = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, B_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, B_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

The inner-coupling matrices are given by,

$$H_1 = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 1 & 1 & -1 \end{bmatrix}, H_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}, H_3 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & -1 & 1 \end{bmatrix}$$

From Figure 5.4,

$$L = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The left eigenvectors of A_1 are $\xi^1_{11} = \begin{bmatrix} -1 & -1 & 1 \end{bmatrix}$, $\xi^1_{12} = \begin{bmatrix} -1 & 1 & 1 \end{bmatrix}$ and the only left eigenvector of A_2 is $\xi^1_{21} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$. We have, $\xi^1_{11} H_1 = \xi^1_{12} H_1 = \xi^1_{21} H_2 = 0$.

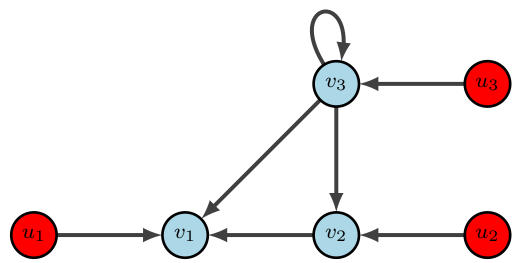

Figure 5.4: Take $\beta_{12} = \beta_{13} = \beta_{23} = \beta_{33} = 1$, otherwise $\beta_{ij} = 0$ and $d_1 = d_2 = d_3 = 1$.

(i) From Figure 5.4, it is clear that all the nodes have external control input.

(ii) $(A_1, B_1), (A_2, B_2)$ and $(A_3 + H_3, B_3)$ are controllable.

Thus, all the conditions of Theorem 5.4 are satisfied. Therefore the given networked system is controllable.

We can also use the Kalaman's rank condition to verify the controllability of the given networked system. The system can be written in the compact form (5.3), where

$$\Omega = \begin{bmatrix} 1 & 2 & 1 & 1 & 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & -1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 & 2 \end{bmatrix} \text{ and } \Psi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

We can see that the controllability matrix $\mathcal{Q}(\Omega, \Psi)$ has rank 9 and hence the given networked system is controllable.

5.5 Conclusions

In Chapter 4, we established a comprehensive set of necessary and sufficient conditions for the controllability of heterogeneous networked systems with identical control input matrices. In this chapter, we broaden our analysis to include heterogeneous systems with non-identical control matrices at each node, and we derive a set of criteria to go with it.

This result has an apparent benefit of being able to pinpoint nodes that require external control inputs in order to change an initially uncontrollable system into a controllable one. Beyond node identification, this result makes it easier to determine the exact control input matrices required for controllability. Theorem 5.1 is a generalization of Theorem 4.4, increasing its scope and applicability over a broader spectrum of systems. The present literature on controllability of heterogeneous networked systems is limited. Kong et al.(Kong et al., 2021) obtained a necessary and sufficient condition for controllability of heterogeneous networked systems having non-identical inner-coupling matrices using the notion of determinant factor and Smith normal form of a matrix. However, the result is computationally demanding and it does not give any information regarding the effect of the system components on controllability of networked systems. Existing studies tell us less about how subsystem dynamics, network topology, and so on affect the controllability of a networked system than ours do, and our results are simple to verify. In addition, controllability results for a more general class of heterogeneous networked systems are obtained over particular network topology, where the inner coupling matrices in each node are distinct.

Chapter 6

Controllability of Networked Systems with Non-linearities

6.1 Introduction

The study of networked systems is critical for comprehending complex interactions and behaviors that emerge in interconnected entities ranging from technological networks to biological systems(Bassett and Sporns, 2017; Farhangi, 2009; Gu et al., 2015; Wang and Chen, 2003; Wuchty, 2014). The prior chapters focused on linear networked systems, providing foundational insights. This chapter, on the other hand, presents a critical extension by integrating non-linear components at each node of the system. This expansion is important because it more properly reflects real-world circumstances in which nonlinear dynamics play a prominent role. The incorporation of non-linearities adds another degree of complexity, making the study of networked systems much more relevant, since it allows for a greater understanding of the intricate dynamics inherent in interconnected systems. This comprehensive investigation adds vital knowledge necessary for tackling real-world difficulties and optimizing the performance of various systems. In this chapter, we consider networked systems with individual nodes having both linear and non-linear parts. There are numerous studies on controllability of stand-alone systems having non-linear components(Joshi and George, 1989; Mirza and Womack, 1971, 1972; Nandakumaran et al., 2017; Vidyasagar, 1972); however, there needs to be more literature in the area of controllability of networked systems having non-linear components. All of these findings are obtained using fixed point theorems. Mainly, we are using the Banach Fixed Point theorem in this Chapter.

The chapter is organized as follows; Controllability problem is formulated in Section 6.2 and in Section 6.3, controllability result for nonlinear networked systems is obtained. Examples are provided to substantiate the result. Conclusions are given in Section 6.4.

6.2 Problem Formulation

Consider a networked linear time invariant system with N nodes, where each node system is of dimension, n. Specifically, the dynamical system corresponding to the node i is described by

$$\dot{x_i}(t) = A_i x_i(t) + \sum_{j=1}^{N} \beta_{ij} H x_j(t) + d_i B_i u_i(t) + f_i(t, x_i(t)), \quad i = 1, 2, \ldots, N \tag{6.1}$$

where $x_i(t) \in \mathbb{R}^n$ is the state vector; $u_i(t) \in \mathbb{R}^m$ is the external control input vector; $A_i \in \mathbb{R}^{n\times n}$ and $B_i \in \mathbb{R}^{n\times m}$ are the state matrix and control input matrix of node i respectively. d_i indicates whether the node i has a control input or not.

$$d_i = \begin{cases} 1, & \text{if node } i \text{ is under control} \\ 0, & \text{otherwise} \end{cases}$$

$\beta_{ij} \in \mathbb{R}$ represents the coupling strength between the nodes i and j with $\beta_{ij} \neq 0$ if there is a communication from node j to node i, but otherwise $\beta_{ij} = 0$, for all $i, j = 1, 2, \ldots, N$. $H \in \mathbb{R}^{n\times n}$ is the inner coupling matrix describing the interactions among the components of $x_j,\ j = 1, 2, \ldots, N$. Let $f_i : [t_0, t_f] \times \mathbb{R}^n \to \mathbb{R}^n,\ i = 1, 2, \ldots, N$ be nonlinear functions such that $t \to f_i(t, .)$ is measurable and $x \to f_i(., x)$ is continuous. We also assume that $f_i, i = 1, 2, \ldots, N$ are Lipschitz continuous with respect to x and is uniformly bounded. That is, there exists $M > 0$ such that $\|f_i(t, x_i(t))\| \leq M_i$ for each i. Let

$$L = [\beta_{ij}] \in \mathbb{R}^{N\times N} \quad \text{and} \quad D = diag\{d_1, d_2, \ldots, d_N\} \tag{6.2}$$

represent the network topology and external input channels of the networked system (6.1), respectively. Denote the whole state of the networked system by $x = \left[x_1^T, \ldots, x_N^T\right]^T$ and the total external control input vector by $u = \left[u_1^T, \ldots, u_N^T\right]^T$. Then by using the Kronecker product the networked system (6.1) can be rewritten in a compact form as

$$\dot{x}(t) = \Omega x(t) + \Psi u(t) + F(t, x(t)) \tag{6.3}$$

with

$$\Omega = A + L \otimes H$$

$$\Psi = blockdiag\{d_1 B_1, \ldots d_N B_N\}$$

and $F(t,x(t)) = \left[f_1(t,x_1(t))^T, \ldots, f_N(t,x_N(t))^T\right]$ where $A = blockdiag\{A_1, \ldots, A_N\}$.

6.3 Controllability of Networked Systems with Non-Linearities

In this section, we consider the controllability problem of networked systems having non-linearities in each individual nodes and linear part of each node is assumed to be controllable.

Suppose that the linear part of (6.3) is controllable. Then by Theorem 1.1, the Controllability Gramian matrix is invertible. That is, the matrix

$$\mathcal{W}(t_0,t_f) = \int_{t_0}^{t_f} \Phi(t_f,\tau)\Psi\Psi^T\Phi^T(t_f,\tau)d\tau$$

is non-singular in the interval $[t_0, t_f]$, where Φ is the state transition matrix corresponding to Ω in (6.3). With initial condition $x(t_0) = x_0$, the solution of the non-linear system can be written as:

$$x(t) = \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,\tau)\left[\Psi u(\tau) + F(\tau,x(\tau))\right]d\tau \tag{6.4}$$

Define a control function

$$\tilde{u}(t) = \Psi^T\Phi^T(t_f,t)\mathcal{W}^{-1}\left[x_1 - \Phi(t_f,t_0)x_0 - \int_{t_0}^{t_f} \Phi(t,\tau)F(\tau,x(\tau))d\tau\right] \tag{6.5}$$

If we substitute, (6.5) in (6.4), we have

$$x(t) = \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,\tau)\Psi\Psi^T\Phi^T(t_f,t)\mathcal{W}^{-1}\left[x_1 - \Phi(t_f,t_0)x_0 - \int_{t_0}^{t_f} \Phi(t,\tau)F(\tau,x(\tau))d\tau\right]d\tau$$
$$+ \int_{t_0}^{t} \Phi(t,\tau)F(\tau,x(\tau))d\tau$$

Then, we can see that

$$x(t_0) = \Phi(t_0,t_0)x_0 + \int_{t_0}^{t_0} \Phi(t_0,\tau)\Psi\Psi^T\Phi^T(t_f,t)\mathcal{W}^{-1}\left[x_1 - \Phi(t_f,t_0)x_0 - \int_{t_0}^{t_f} \Phi(t_0,\tau)F(\tau,x(\tau))d\tau\right]d\tau$$
$$+ \int_{t_0}^{t_0} \Phi(t_0,\tau)F(\tau,x(\tau))d\tau = Ix_0 = x_0$$

as $\Phi(t_0, t_0) = I$. Also,

$$x(t_f) = \Phi(t_f, t_0)x_0 + \int_{t_0}^{t} \Phi(t_f, \tau)\Psi\Psi^T\Phi^T(t_f, t)\mathcal{W}^{-1}\left[x_1 - \Phi(t_f, t_0)x_0 - \int_{t_0}^{t_f} \Phi(t_f, \tau)F(\tau, x(\tau))d\tau\right] d\tau$$

$$+ \int_{t_0}^{t_f} \Phi(t_f, \tau)F(\tau, x(\tau))d\tau$$

$$= \Phi(t_f, t_0)x_0 + x_1 - \Phi(t_f, t_0)x_0 - \int_{t_0}^{t_f} \Phi(t_f, \tau)F(\tau, x(\tau))d\tau + \int_{t_0}^{t_f} \Phi(t_f, \tau)F(\tau, x(\tau))d\tau$$

$$= x_1$$

Now it is enough to show that the operator

$$(\mathcal{K}x)(t) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, \tau)\left[\Psi\tilde{u}(\tau) + F(\tau, x(\tau))\right] d\tau \tag{6.6}$$

has a fixed point, where $\tilde{u}(t)$ is as given in (6.5). We will show that $\mathcal{K}^n$ is a contraction map and use Generalized Banach Contraction Principle. As $f_i(t, x_i(t))$ satisfies Lipschitz condition with respect to x_i for each i, we have that F in (6.6) is Lipschitz with Lipschitz constant α. Let, α_0, β, γ and δ denote bounds for $\|\Phi(t_0, t)\|$, $\left\|\Psi\Psi^T\right\|$, $\left\|\Phi^T(t_f, t)\right\|$, and $\left\|\mathcal{W}^{-1}\right\|$ respectively. Then,

$$\|\mathcal{K}x - \mathcal{K}y\| \leq \left\|\int_{t_0}^{t} \Phi(t, \tau)\Psi\Psi^T\Phi^T(t_f, \tau)W^{-1}\left[\int_{t_0}^{t_f} \Phi(t, \tau)[F(\tau, y(\tau)) - F(\tau, x(\tau))]d\tau\right] d\tau\right\|$$

$$+ \left\|\int_{t_0}^{t} \Phi(t, \tau)[F(\tau, y(\tau)) - F(\tau, x(\tau))]d\tau\right\|$$

$$\leq \int_{t_0}^{t} \|\Phi(t, \tau)\|^2 \left\|\Psi\Psi^T\right\| \left\|\Phi^T(t_f, \tau)\right\| \left\|W^{-1}\right\| \|F(\tau, y(\tau)) - F(\tau, x(\tau))\| (t_f - t_0)d\tau$$

$$+ \int_{t_0}^{t} \|\Phi(t, \tau)\| \|F(\tau, y(\tau)) - F(\tau, x(\tau))\| d\tau$$

$$\leq \left[\alpha\alpha_0^2\beta\gamma\delta(t_f - t_0) + \alpha\alpha_0\right] \parallel x - y \parallel (t - t_0)$$

$$= K \parallel x - y \parallel (t - t_0)$$

where $K = \alpha\alpha_0^2\beta\gamma\delta(t_f - t_0) + \alpha\alpha_0$. Similarly,

$$|(\mathcal{K}^2x)(t) - (\mathcal{K}^2y)(t)| \leq \alpha\alpha_0^2\beta\gamma\delta K \parallel x - y \parallel \frac{(t_f - t_0)^2}{2}(t - t_0) + \alpha\alpha_0 K \parallel x - y \parallel \frac{(t - t_0)^2}{2}$$

Thus

$$\parallel \mathcal{K}^2x - \mathcal{K}^2y \parallel \leq K^2\frac{h^2}{2} \parallel x - y \parallel$$

Proceeding in the same way, for any $n \geq 1$

$$\parallel \mathcal{K}^n x - \mathcal{K}^n y \parallel \leq K^n \frac{h^n}{n!} \parallel x - y \parallel$$

For sufficiently large n, $K^n \frac{h^n}{n!}$ can be made less than 1. Thus $\mathcal{K}^n$ is a contraction. Hence by Generalized Banach Contraction Principle $\mathcal{K}$ has a unique fixed point.

Example 6.1. (Ajayakumar and George, 2023c) Consider a non-linear networked system with 2 individual nodes, where the state matrices and control matrices are given by;

$$A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Let $D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Also, take

$$L = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } H = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

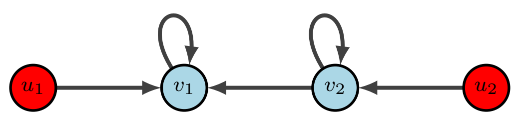

Figure 6.1: Network graph of the given system.

Let the first node be having the non-linearity

$$f_1(t, x_1(t)) = \begin{bmatrix} \frac{t}{k_1} sin\ x_{11}(t) \\ \frac{t}{k_1} cos\ x_{12}(t) \end{bmatrix}$$

and second node be having the non-linearity

$$f_2(t, x_2(t)) = \begin{bmatrix} \frac{t^2}{k_2} |x_{21}(t)| \\ \frac{t^2}{k_2} cos\ x_{22}(t) \end{bmatrix}$$

where $k_1, k_2 > 0$ are constants. Now,

$$\begin{aligned}\|f_1(t,x_1(t)) - f_1(t,x_2(t))\|^2 &= \max_{0\le t\le 1}\left(\frac{t}{k_1}\right)^2\left[(sinx_{11}(t) - siny_{11}(t))^2 + (cosx_{11}(t) - cosy_{11}(t))^2\right]\\ &\le \left(\frac{1}{k_1}\right)^2 \max_{0\le t\le 1}\left[(sinx_{11}(t) - siny_{11}(t))^2\right] + \\ &\quad \left(\frac{1}{k_1}\right)^2 \max_{0\le t\le 1}\left[(cosx_{11}(t) - cosy_{11}(t))^2\right]\\ &\le \left(\frac{1}{k_1}\right)^2 \max_{0\le t\le 1}\left[(x_{11}(t) - y_{11}(t))^2 + (x_{12}(t) - y_{12}(t))^2\right]\\ &= \left(\frac{1}{k_1}\right)^2 \|x_1(t) - y_1(t)\|^2\end{aligned}$$

Thus f_1 satisfy Lipschitz condition with respect to x_1 in the interval $[0,1]$ with Lipschitz constant $\frac{1}{k_1}$. Similarly, we can prove that f_2 satisfy Lipschitz condition with respect to x_2 in the interval $[0,1]$ with Lipschitz constant $\frac{1}{k_2}$. Then, $F(t,x(t))$ satisfy Lipschitz condition with respect to x in the interval $[0,1]$ with Lipschitz constant $\alpha = max\left\{\frac{1}{k_1}, \frac{1}{k_2}\right\}$. The system can be written in the compact form as

$$\dot{x}(t) = \begin{bmatrix}1 & 1 & 0 & 1\\ 1 & 1 & 0 & 0\\ 0 & 0 & 1 & 2\\ 0 & 0 & 0 & 1\end{bmatrix} x(t) + \begin{bmatrix}1 & 0\\ 0 & 0\\ 0 & 0\\ 0 & 1\end{bmatrix} u(t) + \begin{bmatrix}\frac{t}{k_1} sin\, x_{11}(t)\\ \frac{t}{k_1} cos\, x_{12}(t)\\ \frac{t^2}{k_2}|x_{21}(t)|\\ \frac{t^2}{k_2} cos\, x_{22}(t)\end{bmatrix}$$

We can see that the controllability matrix,

$$\mathcal{Q}(\Omega,\Psi) = [\Psi \mid \Omega\Psi \mid \Omega^2\Psi \mid \Omega^3\Psi] = \begin{bmatrix}1 & 0 & 1 & 1 & 2 & 2 & 4 & 4\\ 0 & 0 & 1 & 0 & 2 & 1 & 4 & 3\\ 0 & 0 & 0 & 2 & 0 & 4 & 0 & 6\\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1\end{bmatrix}$$

has rank 4. Thus by Kalman's rank condition the linear part of the given system is controllable. The state transition matrix is

$$e^{\Omega t} = \begin{bmatrix}\frac{1}{2}\left(e^{2t}+1\right) & \frac{1}{2}\left(e^{2t}-1\right) & 0 & \frac{1}{2}\left(e^{2t}-1\right)\\ \frac{1}{2}\left(e^{2t}-1\right) & \frac{1}{2}\left(e^{2t}+1\right) & 0 & \frac{1}{2}\left(e^{2t}+1\right) - e^t\\ 0 & 0 & e^t & 2te^t\\ 0 & 0 & 0 & e^t\end{bmatrix}$$

The controllability gramian matrix, $\mathcal{W}(0,1)$ and its inverse are given by,

$$\mathcal{W}(0,1) = \begin{bmatrix} 7.1998 & 3.8780 & 3.5746 & 2.3218 \\ 3.8780 & 2.3142 & 1.3800 & 0.8455 \\ 3.5746 & 1.3800 & 6.3891 & 4.1945 \\ 2.3218 & 0.8455 & 4.1945 & 3.1945 \end{bmatrix}$$

and

$$\mathcal{W}^{-1}(0,1) = \begin{bmatrix} 2.4722 & -3.8201 & -0.3059 & -0.3841 \\ -3.8201 & 6.4010 & 0.3202 & 0.6619 \\ -0.3059 & 0.3202 & 1.2189 & -1.4630 \\ -0.3841 & 0.6619 & -1.4630 & 2.3379 \end{bmatrix}$$

Also, $e^{\Omega^T t}$ equals

$$\begin{bmatrix} \frac{1}{2}\left(e^{2t}+1\right) & \frac{1}{2}\left(e^{2t}-1\right) & 0 & 0 \\ \frac{1}{2}\left(e^{2t}-1\right) & \frac{1}{2}\left(e^{2t}+1\right) & 0 & 0 \\ 0 & 0 & e^t & 0 \\ \frac{1}{2}\left(e^{2t}-1\right) & \frac{1}{2}\left(e^{2t}+1\right)-e^t & 2te^t & e^t \end{bmatrix}$$

and

$$\Psi\Psi^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Then we can find the value of K. We have

$$K = \alpha\alpha_0^2\beta\gamma\delta(t_f - t_0) + \alpha\alpha_0 = \alpha\left[\alpha_0^2\beta\gamma\delta(t_f - t_0) + \alpha_0\right]$$

Here

$$\alpha_0 = 12.8256$$

$$\beta = 1$$

$$\gamma = 10.5835$$

and

$$\delta = 11.2032$$

We can choose k_1 and k_2 in such a way that $\mathcal{K}^n$ is a contraction for sufficiently large n. Hence $\mathcal{K}$ has a unique fixed point which proves the controllability of the given non-linear system.

6.4 Conclusions

In this Chapter, *Generalized Banach Contraction Principle* is used to derive a sufficient condition for obtaining the controllability of a non-linear networked system with nodes that include both linear and non-linear components. It is specifically assumed that the linear part of the networked system is controllable. The study makes a substantial contribution to the subject by proving that the controllability of the networked system is assured when the non-linear components within each node adhere to Lipschitz conditions and are uniformly bounded. This important result not only validates controllability but also allows for the evaluation of the control function for the given system using the iterative scheme provided by the Banach Contraction Principle. The study contains illustrative example that highlight and validate the derived conditions in practical networked environments to improve the clarity and application of the theoretical results.

Chapter 7

Generic Controllability of Networks with Non-Identical SISO Dynamical Nodes

7.1 Introduction

Controllability research advanced to complex networks, reflecting the idea that the real-world systems demanded sophisticated modeling. The studies of individual systems connected together can be traced back to the work done by Gilbert (Gilbert, 1963) which was then followed by many others as the controllability and observability of interconnected systems became a topic of interest (Callier and Nahum, 1975; Chen and Desoer, 1967; Davison, 1977; Fuhrmann, 1975). Large-scale networks posed difficulties in getting precise parameter values for system dynamics, motivating the use of structural controllability in order to overcome this limitation. Glover et al.(Glover and Silverman, 1976) and Shields et al.(Shields and Pearson, 1976) contributed to the extensive research of these situations as they progressed from single-input to multi-input systems. Linnemann(Linnemann, 1986) simplified Lin's structural controllability theorem, which he first proposed in 1974. Mayeda et al.(Li et al., 2015) established strong structural controllability, which ensures system controllability for every parameter value, with Hosoe et al.(Hosoe and Matsumoto, 1979) strengthening the algebraic criteria. Mayeda(Mayeda, 1981) investigated graph-theoretic interpretations, which provided insights into the full rank condition of structured matrices. The presence of complex networks in numerous scientific and technical domains has resulted in an increase in study on the controllability of networks of dynamical systems(Blackhall and Hill, 2010; Chapman and Mesbahi, 2013; Liu et al., 2013; Pequito et al., 2015; Xue and Roy, 2019; Zhang et al., 2021). New tools and approaches were introduced, and structural controllability of networks is still being researched. The timeline of structural system study can be tracked through the works of Dion et al.(Dion et al., 2003), Ramos et al.(Ramos et al., 2020), and Xiang et al.(Xiang et al., 2019a).

Commault et al.(Commault and Kibangou, 2019) established the concept of generic controllability, in which system matrices for each node stay constant but link weights across nodes are unknown, providing a new dimension to controllability research. This is a relatively new concept in the area of controllability of inter connected systems. However, the conditions obtained are structural as they are based on the composition of the network graph. Commault et al.(Commault and Kibangou, 2019) examined the generic controllability of inter connected systems where individual systems having same dynamics are connected together and obtained a necessary and sufficient condition for generic controllability of networked systems.

In this Chapter, we prove that the conditions derived by Commault et al.(2019)(Commault and Kibangou, 2019) for the generic controllability of homogeneous networked systems are necessary for the generic controllability of networked systems having heterogeneous dynamics. The sections are arranged as follows: Formulation of the controllability problem is given in section 7.2. Some necessary conditions for generic controllability of networked systems is obtained in Section 7.3. The obtained results are illustrated with examples. Concluding remarks and future works are stated in section 7.4.

7.2 Problem Formulation

Consider a networked system, with N state nodes and m control nodes interacting via weighted directed connections. The weighted directed graph $G(\mathcal{N}) = (V_{\mathcal{N}}, E_{\mathcal{N}})$, called the network graph can be used to represent the network, $\mathcal{N}$. The vertex set of the network graph is given by, $V_{\mathcal{N}} = \{v_1, v_2, \ldots, v_N\} \cup \{u_1, u_2, \ldots, u_m\}$, where $v_i's$ and $u_i's$ represent the state nodes and control nodes, respectively. The directed connections between the nodes is represented by the edge set $E_{\mathcal{N}}$. Edge weights assigned to the network graph quantifies the strength of the communication between the individual nodes.

The node v_i represents a dynamical system with n states, a scalar input w_i, and a scalar output y_i. The dynamics of the node v_i is given by

$$\begin{aligned} \dot{x}_i(t) &= A_i x_i(t) + B_i w_i(t) \\ y_i(t) &= C_i x_i(t) \end{aligned} \tag{7.1}$$

where $A_i \in \mathbb{R}^{n\times n}$ for each i and B_i(respectively, C_i) is a $n-$ dimensional column vector (respectively, a row vector) for each i. The dynamic state of each node is denoted by the matrices (A_i, B_i, C_i).

Combining the state space model representing the dynamics of each node with the composition of the network graph, we get a global system $\sum_{\mathcal{N}}$ of state space dimension Nn and m control inputs. The input signal for the node i is given by the weighted combination of control signals in line with the network graph

$$w_i(t) = \sum_{i=1}^{N} \beta_{ij} y_j(t) + \sum_{l=1}^{m} \delta_{il} u_l(t) \tag{7.2}$$

where β_{ij} represents the connection strength of the link from node v_j to node v_i, δ_{il} represents the connection strength of the link from control node u_l to the state node v_i. β_{ij} and δ_{il} becomes zero when there is no edge in the network graph between the state nodes or from a control node to a state node respectively. Let $L = [\beta_{ij}]_{N\times N}, i = 1, 2, \ldots, N, j = 1, 2, \ldots, N$ and $\Delta = [\delta_{il}]_{N\times m}, i = 1, 2, \ldots, N, l = 1, 2, \ldots, m$ represent network topology.

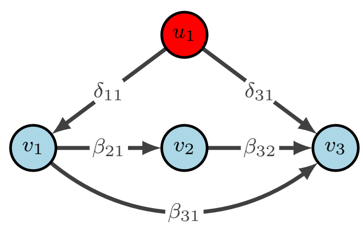

Figure 7.1: Example of a networked system with 3 state nodes and one control node.

Then the compact form of $\sum_{\mathcal{N}}$ is given by

$$\sum_{\mathcal{N}} : \dot{x}(t) = \Omega x(t) + \Psi u(t) \tag{7.3}$$

where $x(t) = (x_1(t), \ldots, x_m(t))^T$ and $u(t) = (u_1(t), \ldots, u_m(t))^T$, with $(.)^T$ indicates the transpose of a matrix. The matrices Ω and Ψ representing the state and control matrices of $\sum_{\mathcal{N}}$, respectively, have dimensions $Nn \times Nn$ and $Nn \times m$. They are of the following form:

$$\Omega = \begin{bmatrix} A_1 + \beta_{11} B_1 C_1 & \beta_{12} B_1 C_2 & \ldots & \beta_{1N} B_1 C_N \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{N1} B_N C_1 & \beta_{N2} B_N C_2 & \ldots & A_N + \beta_{NN} B_N C_N \end{bmatrix}$$

and

$$\Psi = \begin{bmatrix} \delta_{11} B_1 & \delta_{12} B_1 & \ldots & \delta_{1m} B_1 \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{N1} B_N & \delta_{N2} B_N & \ldots & \delta_{Nm} B_N \end{bmatrix}$$

In this work, our aim is to analyse the controllability of $\sum_{\mathcal{N}}$, using the dynamics of the individual systems. That is, using the matrices $(A_i, B_i, C_i)'s$, and structure of the networked system. The matrices $(A_i, B_i, C_i)'s$ are assumed to be exact and known, but the network communication strengths are not fixed precisely. That is, we know whether the entries are zero or non-zero, but we do not know the exact parameter values.

7.3 Necessary Conditions for Generic Controllability of Heterogeneous Networked Systems

In (Commault and Kibangou, 2019), Commault et al. give the following set of conditions which are necessary and sufficient for the generic controllability of interconnected systems with identical dynamical nodes.

Theorem 7.1. *(Commault and Kibangou, 2019) Consider a network $\mathcal{N}$ with N internal nodes, m control nodes with $N > m$, and its graph $G(\mathcal{N})$. Assume that all nodes are identical, SISO, n^{th}-order dynamical systems defined by matrices A, B, C. The global system $\Sigma_{\mathcal{N}}$ is generically controllable if and only if the following conditions hold:*

(i) *The pair (A, B) is controllable.*

(ii) *The pair (C, A) is observable.*

(iii) *The graph $G(\mathcal{N})$ is control-connected.*

(iv) *The internal nodes of $G(\mathcal{N})$ can be covered by a disjoint set of stems and cycles.*

We will show that the first three conditions in Theorem 7.1 are necessary for the generic controllability of networked systems with non-identical nodes.

Theorem 7.2. *(Ajayakumar and George, 2023d) If the pair (A_i, B_i) is not controllable for some i, say i_0, then the global system is not generic controllable.*

Proof: Suppose that (A_i, B_i) is not controllable for some i, say i_0 , then by PBH criterion there exists a scalar λ and a row vector v such that

$$v\left(A_{i_0} - \lambda I\right) = 0 \text{ and } vB_{i_0} = 0$$

Now consider the vector $e_{i_0} \otimes v$, where $e_{i_0} \in \mathbb{R}^{1\times N}$ with i_0^{th} entry 1 and all other entries zero. Then

$$\left(e_{i_0} \otimes v\right)(\Omega - \lambda I) = 0 \quad \text{and} \quad \left(e_{i_0} \otimes v\right) G = 0$$

That is, (Ω, Ψ) is not controllable.

Example 7.1. Consider a heterogeneous networked system with two nodes, with state matrices $A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and control matrices $B_1 = B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The output matrices are given by $C_1 = C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}$. Take,

$$L = \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix} \text{ and } \Delta = \begin{bmatrix} \delta_{11} \\ \delta_{21} \end{bmatrix}$$

Here, (A_1, B_1) is not controllable as the controllability matrix,

$$\mathcal{Q}(A_1, B_1) = [B_1 \mid A_1 B_1] = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

has rank 1. (A_2, B_2) is controllable as the controllability matrix,

$$\mathcal{Q}(A_2, B_2) = [B_2 \mid A_2 B_2] = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

has rank 2. By Theorem7.2, the given system is not controllable as (A_1, B_1) is not controllable. We can verify this using PBH rank criterion. The given system can be written in the compact form (7.3) with

$$\Omega = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1+\beta_{11} & \beta_{12} & 0 \\ 0 & 0 & 1 & 1 \\ 0 & \beta_{21} & \beta_{22} & 1 \end{bmatrix} \text{ and } \Psi = \begin{bmatrix} 0 \\ \delta_{11} \\ 0 \\ \delta_{12} \end{bmatrix}$$

As the matrix

$$[\Omega - I, \Psi] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & \beta_{11} & \beta_{12} & 0 & \delta_{11} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & \beta_{21} & \beta_{22} & 0 & \delta_{21} \end{bmatrix}$$

has rank at-most 3 only for any values of β_{ij} and δ_{il}, by PBH rank criterion the given networked system is not controllable.

Theorem 7.3. *(Ajayakumar and George, 2023d) If $N > m$, for the global system to be generic controllable atleast one of the pairs $(A_i, C_i)\,, i = 1, 2, \ldots, N$ must be observable.*

Proof: Suppose that (A_i, C_i) is not observable for all $i = 1, 2, \ldots, N$. Then by PBH criteria there exists a scalar λ and a column vector v_i such that $(A_i - \lambda I)\, v_i = 0$ and $C_i v_i = 0$. Now consider the vector $(e_i \otimes v_i)$, where $e_i \in \mathbb{R}^{1 \times N}$ with ith entry 1 and all other entries zero. Then

$$(\Omega - \lambda I)\,(e_i \otimes v_i) = 0$$

Then

$$rank\,[\Omega - \lambda I, \Psi] \leq N(n-1) + m < Nn$$

Therefore, (Ω, Ψ) is not controllable.

Example 7.2. Consider a heterogeneous networked system with two nodes, with state matrices $A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and control matrices $B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The output matrices are given by $C_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}$ and $C_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}$. Take,

$$L = \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix} \text{ and} \Delta = \begin{bmatrix} \delta_{11} \\ \delta_{21} \end{bmatrix}$$

Here both (A_1, C_1) and (A_2, C_2) are not observable as both observability matrices

$$\mathcal{O}(A_1, C_1) = \begin{bmatrix} C_1 \\ C_1 A_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

and

$$\mathcal{O}(A_2, C_2) = \begin{bmatrix} C_2 \\ C_2 A_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

have rank 1. Also $N > m$. Therefore by Theorem 7.3 the given system is not controllable. We can verify this using PBH rank criterion. The given system can be written in the compact form (7.3), with

$$\Omega = \begin{bmatrix} 1+\beta_{11} & 0 & 0 & \beta_{12} \\ 1+\beta_{11} & 1 & 0 & \beta_{12} \\ 0 & 0 & 1 & 1 \\ \beta_{21} & 0 & 0 & 1+\beta_{22} \end{bmatrix} \text{ and } \Psi = \begin{bmatrix} \delta_{11} \\ \delta_{11} \\ 0 \\ \delta_{12} \end{bmatrix}$$

As the matrix

$$[\Omega - I, \Psi] = \begin{bmatrix} \beta_{11} & 0 & 0 & \beta_{12} & \delta_{11} \\ 1+\beta_{11} & 0 & 0 & \beta_{12} & \delta_{11} \\ 0 & 0 & 0 & 1 & 0 \\ \beta_{21} & 0 & 0 & \beta_{22} & \delta_{21} \end{bmatrix}$$

has rank at-most 3 only for any values of β_{ij} and δ_{il}, by PBH rank criterion the given networked system is not controllable.

Theorem 7.4. *(Ajayakumar and George, 2023d) If the graph $G(\mathcal{N})$ is not control connected, then the global system is not generic controllable.*

Proof: Suppose that $G(\mathcal{N})$ is not control connected. Rearrange the nodes so that the first k nodes represent the non control connected nodes. Then the matrices L and Δ can be expressed as

$$L = \begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix} \text{ and } \Delta = \begin{bmatrix} 0_{k \times k} \\ \Delta_2 \end{bmatrix}$$

where L_{11} is a $k \times k$ matrix. Then Ω and Ψ are of the form

$$\Omega = \begin{bmatrix} \Omega_{11} & 0 \\ \Omega_{21} & \Omega_{22} \end{bmatrix} \text{ and } \Psi = \begin{bmatrix} 0_{kn \times kn} \\ \Psi \end{bmatrix}$$

where Ω_{11} is a $kn \times kn$ matrix. Now for any left eigenvector v of Ω_{11}, $\tilde{v} = \begin{bmatrix} v & 0_{n(N-k)} \end{bmatrix}$ is a left eigenvector of Ω with $v\Psi = 0$. Therefore, (Ω, Ψ) is not controllable.

Example 7.3. Consider a heterogeneous networked system with two nodes, with state matrices $A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and control matrices $B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The output matrices are given by $C_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}$ and $C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}$. Take,

$$L = \begin{bmatrix} 0 & \beta_{12} \\ 0 & 0 \end{bmatrix} \text{ and } \Delta = \begin{bmatrix} \delta_{11} \\ 0 \end{bmatrix}$$

Here (A_1, B_1) and (A_2, B_2) are controllable as both controllability matrices

$$\mathcal{Q}(A_1, B_1) = [B_1 \mid A_1 B_1] = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

and

$$\mathcal{Q}(A_2, B_2) = [B_2 \mid A_2 B_2] = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

have rank 2. Also, both (A_1, C_1) and (A_2, C_2) are observable as both observability matrices

$$\mathcal{O}(A_1, C_1) = \begin{bmatrix} C_1 \\ C_1 A_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

and

$$\mathcal{O}(A_2, C_2) = \begin{bmatrix} C_2 \\ C_2 A_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

have rank 2. $G(\mathcal{N})$ is not control connected as there does not exist a control-state path from u_1 to

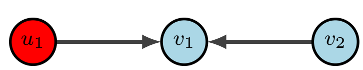

Figure 7.2: Clearly, $G(\mathcal{N})$ is not control connected.

v_2. Then, by Theorem 7.4 the given system is not controllable. We can verify this using PBH rank criterion. The given system can be written in the compact form (7.3), with

$$\Omega = \begin{bmatrix} 1 & 0 & \beta_{12} & 0 \\ 1 & 1 & \beta_{12} & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } \Psi = \begin{bmatrix} \delta_{11} \\ \delta_{11} \\ 0 \\ 0 \end{bmatrix}$$

As the matrix

$$[\Omega - I, \Psi] = \begin{bmatrix} 0 & 0 & \beta_{12} & 0 & \delta_{11} \\ 1 & 0 & \beta_{12} & 0 & \delta_{11} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

has rank at-most 3 only for any values of β_{ij} and δ_{il}, by PBH rank criterion the given networked system is not controllable.

7.4 Conclusions

In this Chapter, generic controllability of interconnected linear systems with heterogeneous dynamics is studied. It has been shown that some of the necessary conditions for the generic controllability of homogeneous networked systems obtained by Commault et al.(Commault and Kibangou, 2019) like the controllability of the individual nodes, observability of the individual nodes and the control connectedness of the netork graph stay necessary for heterogeneous networked systems also. The advantage of the derived results is that we can discuss the controllability of a networked system without having full knowledge of the network topology. The obtained results are supplemented with suitable examples.

Chapter 8

Summary of the Thesis and Future Work

In this section, we provide the significant contributions of the thesis as well as future plan of research based on the current work. Controllability studies have grown in popularity in recent decades, with the notion being proposed by R.E. Kalman in the latter half of the twentieth century(Kalman, 1960, 1962). According to Kalman, controllability is the ability of a dynamical system to reach a desired final state from any arbitrary initial state within a finite time period. Initially focused on single higher-dimensional systems with known parameter values, practical applications demonstrated the necessity for a broader framework due to variances or inaccurate knowledge of these parameters. The notion of structural controllability by C.T. Lin(Lin, 1974, 1977) addressed this difficulty by highlighting the importance of network structures in the controllability of a dynamical system. In recent years, controllability research expanded to complex networks, reflecting the idea that real-world systems necessitate sophisticated modeling. The objective of this thesis is to study the controllability and observability of such networked systems with a special focus on the factors such as individual node dynamics, network topology and inner-coupling matrices. The contributions of the thesis are summarized as follows.

In **Chapter 3**, the notion of controllability for the homogeneous networked system

$$\dot{x_i}(t) = Ax_i(t) + \sum_{j=1}^{N} \beta_{ij} HCx_j(t) + d_i Bu_i(t), \quad i = 1, 2, \dots, N \tag{8.1}$$

is introduced, where, $x_i(t) \in \mathbb{R}^n$ is a state vector of the i^{th} node; $u_i(t) \in \mathbb{R}^m$ is an external control input vector applied to the i^{th} node; $y_i(t) \in \mathbb{R}^n$ is an output vector of the i^{th} node; $A \in \mathbb{R}^{n\times n}$ is the state matrix, $B \in \mathbb{R}^{n\times m}$ is the input matrix and $C \in \mathbb{R}^{m\times n}$ is the output matrix of node i. If node i under external control, then $d_i = 1$, otherwise $d_i = 0$. $\beta_{ij} \in \mathbb{R}$ represents the communication strength between the nodes i and j. A communication from node j to node i ensures that $\beta_{ij} \neq 0$, otherwise $\beta_{ij} = 0$, for all $i, j = 1, 2, \dots, N$. The inner coupling matrix describing the interconnections among the components $x_j, j = 1, 2, \dots, N$ is denoted by $H \in \mathbb{R}^{n\times m}$.

Let $L = [\beta_{ij}] \in \mathbb{R}^{N\times N}$ represent the network topology and $D = diag\{d_1, d_2, \dots, d_N\}$, the external input channels of the networked system (8.1). Also, let $X = \left[x_1^T, \ldots, x_N^T\right]^T$ denote the network state and $U = \left[u_1^T, \ldots, u_N^T\right]^T$, the total external control of the networked system. Using Kronecker product, the homogeneous networked system (8.1) can be rewritten in the compact form as

$$\dot{X}(t) = \Omega X(t) + \Psi U(t) \tag{8.2}$$

with

$$\Omega = I_N \otimes A + L \otimes HC \text{ and } \Psi = D \otimes B \tag{8.3}$$

Wang et al. (Wang et al., 2016b) derived the following necessary and sufficient condition for the controllability of the homogeneous networked system (8.2)-(8.3).

Theorem 8.1. *(Wang et al., 2016b) The networked system* (8.2)-(8.3) *is controllable if and only if, for any complex number s, the matrix solution $F \in \mathbb{C}^{N\times n}$ of the simultaneous equations*

$$\begin{cases} D^T F B = 0 \\ L^T F H C = F(sI - A) \end{cases} \tag{8.4}$$

is $F = 0$.

Also, the following necessary conditions for the controllability of networked system (8.2)-(8.3) were obtained, which indicates the effect of the components of the networked system on controllability of the networked system.

Theorem 8.2. *(Wang et al., 2016b) Suppose that the networked system* (8.2)-(8.3) *is controllable.*

(a) If there exists one node without incoming edges, it is necessary that (A, B) is controllable and moreover an external control input is applied onto this node which has no incoming edges.

(b) If there exists one node without external control inputs, it is necessary that (A, HC) is controllable.

(c) If the number of individual nodes is N and the number of nodes with external control is m with $N > m.rank(B)$, then it is necessary that (A, C) is observable.

(d) (L, D) is a controllable pair.

Later, Wang P. et al.(2017)(Wang et al., 2017b) and Xiang et al.(Xiang et al., 2019b) tried to extend the results by Wang et al.(2016) (Wang et al., 2016b) for the heterogeneous networked system

$$\dot{x_i}(t) = A_i x_i(t) + \sum_{j=1}^{N} \beta_{ij} H C_j x_j(t) + d_i B_i u_i(t), \quad i = 1, 2, \ldots, N \tag{8.5}$$

The above system can be written in the compact form as

$$\dot{X}(t) = \Omega X(t) + \Psi U(t) \tag{8.6}$$

where,

$$\Omega = \Lambda + \Gamma \text{ and } \Psi = diag\{d_1 B_1, \ldots, d_N B_N\} \tag{8.7}$$

where,

$$\Lambda = diag\{A_1, \ldots, A_N\} \text{ and } \Gamma = \left[\beta_{ij} H C_j\right] \in \mathbb{R}^{nN \times nN} \tag{8.8}$$

Xiang et al.(Xiang et al., 2019b) derived that for the controllability of certain networked systems the observability of the node system is necessary.

Theorem 8.3. *(Xiang et al., 2019b) Suppose* $N > \Sigma_{i=1}^{\tilde{m}} rank(B_i)$*($\tilde{m}$ is the number of external control inputs),* $A_1, \ldots, A_N$ *are similar to each other, and there exists* $k_i \neq 0, i = 1, 2, \ldots, N$*, such that* $k_1 C_1 = k_2 C_2 = \ldots = k_N C_N$*. For the heterogeneous networked system* (8.6)-(8.7) *to be controllable, it is necessary that* (A_i, C_i) *is observable for* $i = 1, 2, \ldots, N$*.*

We have given an example to show that this result is not true in general and restated the theorem as follows.

Theorem 8.4. *(Ajayakumar and George, 2022b) Suppose* $N > \sum_{i=1}^{\tilde{m}} rank(B_i)$*. Let* $A_1, A_2, \ldots, A_N$ *be similar to each other, i.e., for each* A_i *there exists an invertible matrix* P_i^k *such that* $\left(P_i^k\right)^{-1} A_i P_i^k = A_k$*, for all* $i = 1, 2, \ldots, N$ *and* $k = 1, 2, \ldots, N$*. Also there exists* $k_i \neq 0, i = 1, 2, \ldots, N$*, such that* $k_1 C_1 = \cdots = k_N C_N$*. For the controllability of the heterogeneous networked system* (8.6) - (8.7)*, the observability of* (A_i, C_i) *is necessary for all* $i = 1, 2, \ldots, N$*, if the matrix* P_i^k *commutes with* C_i*.*

We have also given certain situations where observability of the networked system is necessary for the controllability of (8.6) - (8.7) in **Chapter 3**.

In **Chapter 4**, we consider heterogeneous networked systems of the form:

$$\dot{x}_i(t) = A_i x_i(t) + \sum_{j=1}^{N} \beta_{ij} H x_j(t) + d_i B u_i(t), \quad i = 1, 2, \ldots, N \tag{8.9}$$

the networked system (8.9) can be reduced into the following compact form:

$$\dot{X}(t) = \Omega X(t) + \Psi U(t) \tag{8.10}$$

where,

$$\Omega = \mathcal{A} + L \otimes H \text{ and } \Psi = D \otimes B \tag{8.11}$$

and $\mathcal{A} = blockdiag\{A_1, A_2, \ldots, A_N\}$. If the state node matrices $A_1, A_2, \ldots, A_N$ are identical, that is, $A_i = A,\ i = 1, 2, \ldots, N$, then the system (8.9) becomes a homogeneous networked system of the form (8.1) with the output matrix $C = I$. Hao et al.(Hao et al., 2018) studied the controllability of such systems by characterizing the eigenvalues and eigenvectors of the component matrices as follows:

Theorem 8.5. *(Hao et al., 2018) Assume that is L diagonalizable with the set of the eigenvalues $\sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_N\}$. Let $M_i = \{\mu_i^1, \mu_i^2, \ldots, \mu_i^{q_i}\}$ be the set of the eigenvalues of $A + \lambda_i H,\ i = 1, 2, \ldots, N$. Then $\sigma(\Omega) = \left\{\mu_1^1, \mu_1^2, \ldots, \mu_1^{q_i}, \ldots, \mu_N^1, \mu_N^2, \ldots, \mu_N^{q_i}\right\}$. Moreover, the left eigenvectors of Ω associated with μ_i^j are $t_i \otimes \xi_{ij}^1, t_i \otimes \xi_{ij}^2, \ldots, t_i \otimes \xi_{ij}^{\gamma_{ij}}$ where t_i is the left eigenvector of L corresponding to eigenvalue λ_i: $\gamma_{ij} \geq 1$ is the geometric multiplicity of μ_i^j for $A + \lambda_i H$; $\xi_{ij}^k (k = 1, \ldots, \gamma_{ij})$ are the left eigenvectors of $A + \lambda_i H$ corresponding to $\mu_i^j, j = 1, 2, \ldots, q_i, i = 1, 2, \ldots, N$.*

Using this result and PBH eigenvector test, Hao et al.(Hao et al., 2018) derived the following controllability result for the homogeneous networked system (8.1) with $C = I$.

Theorem 8.6. *(Hao et al., 2018) Consider a homogeneous networked system with a diagonalizable network topology matrix L. Let $\sigma(L) = \{\lambda_1, \lambda_2, \ldots, \lambda_N\}$. Then the networked system* (8.1) *with output matrix $C = I$ is controllable if and only if the following conditions are satisfied.*

(i) (L, D) *is controllable;*

(ii) $(A + \lambda_i H, B)$ *is controllable, for $i = 1, 2, \cdots, N$; and*

(iii) If matrices $A+\lambda_{i_1} H, \ldots, A+\lambda_{i_p} H$ $(\lambda_{i_k} \in \sigma(L),\ for\ k = 1, \ldots, p,\ p > 1)$ have a common eigenvalue ρ, then $(t_{i_1} D)\otimes(\xi_{i_1}^1 B), \ldots, (t_{i_1} D)\otimes(\xi_{i_1}^{\gamma_{i_1}} B), \ldots, (t_{i_p} D)\otimes(\xi_{i_p}^1 B), \ldots, (t_{i_p} D)\otimes(\xi_{i_p}^{\gamma_{i_p}} B)$ are linearly independent, where t_{i_k} is the left eigenvector of L corresponding to the eigenvalue λ_{i_k}; $\gamma_{i_k} \geq 1$ is the geometric multiplicity of ρ for $A + \lambda_{i_k} H$; $\xi_{i_k}^l (l = 1, \ldots, \gamma_{i_k})$ are the left eigenvectors of $A + \lambda_{i_k} H$ corresponding to $\rho, k = 1, \ldots, p$.

We extended the result by Hao et al.(Hao et al., 2018) to the networked systems of the form (8.9), where the state matrices are distinct in each node, however the control matrices are the same in each node. We proceeded by characterizing the eigenvectors of the state matrix of the heterogeneous system (8.10) as follows:

Theorem 8.7. *(Ajayakumar and George, 2023b) Let T be the triangulizing matrix for the network topology matrix L and suppose $T \otimes I$ commutes with $\mathcal{A}$. Let (μ_i^j, ξ_{ij}^k) denotes the left eigenpair of $A_i + \lambda_i H$. Then the following statements hold true.*

(i) The eigenspectrum of Ω is the union of eigenspectrum of $A_i + \lambda_i H$, where, $i = 1, 2, \ldots, N$. That is, $\sigma(\Omega) = \cup_{i=1}^N \sigma(A_i + \lambda_i H) = \{\mu_1^1, \ldots, \mu_1^{q_1}, \ldots, \mu_N^1, \ldots, \mu_N^{q_N}\}$.

(ii) *If J is a diagonal matrix, then $e_iT \otimes \xi_{ij}^k, k = 1, \dots, \gamma_{ij}$ are the left eigenvectors of Ω corresponding to the eigenvalue $\mu_i^j, j = 1, \dots, q_i, i = 1, \dots, N$, where $\{e_i : i = 1, 2, \dots, N\}$ is the canonical basis for $\mathbb{R}^N$.*

(iii) *If J contains a Jordan block of order $l \geq 2$ for some eigenvalue λ_{i_0} of L with $\xi_{ij}^k H = 0$ for all $i = i_0, i_0 + 1, \dots, i_0 + l - 1, j = 1, 2, \dots, q_i, k = 1, 2, \dots, \gamma_{ij}$, then $e_iT \otimes \xi_{ij}^k, k = 1, \dots, \gamma_{ij}$ are the left eigenvectors of Ω corresponding to the eigenvalue $\mu_i^j, i = 1, 2, \dots, N, j = 1, 2, \dots, q_i$.*

We used the PBH eigenvector test to derive the following controllability result for the heterogeneous system (8.10).

Theorem 8.8. *(Ajayakumar and George, 2023b) Let T be a non-singular matrix triagularizing matrix L such that $T \otimes I$ commutes with $\mathcal{A}$. If J contains a Jordan block of order $l \geq 2$ corresponding to the eigenvalue λ_{i_0} of L, then assume that $\xi_{ij}^k H = 0$ for all $i = i_0, i_0 + 1, \dots, i_0 + l - 1, j = 1, 2, \dots, q_i, k = 1, 2, \dots, \gamma_{ij}$, where $\xi_{ij}^k, i = 1, 2, \dots, N, j = 1, 2, \dots, q_i, k = 1, 2, \dots, \gamma_{ij}$ are the left eigenvectors of $A_i + \lambda_i H$ corresponding to the eigenvalues $\mu_i^j, i = 1, 2, \dots, N, j = 1, 2, \dots, q_i$. Then the networked system* (8.10) *is controllable if and only if*

(i) *$e_iTD \neq 0$ for all $i = 1, \dots, N$*

(ii) *$(A_i + \lambda_i H, B)$ is controllable, for $i = 1, 2, \cdots, N$; and*

(iii) *If matrices $A_{i_1} + \lambda_{i_1} H, A_{i_2} + \lambda_{i_2} H, \dots, A_{i_p} + \lambda_{i_p} H (\lambda_{i_k} \in \sigma(L), k = 1, \dots, p, \ where\ p > 1)$ have a common eigenvalue ρ, then $(e_{i_1}TD) \otimes (\xi_{i_1}^1 B), \cdots, (e_{i_1}TD) \otimes (\xi_{i_1}^{\gamma_{i_1}} B), \dots, (e_{i_p}TD) \otimes (\xi_{i_p}^1 B), \dots, (e_{i_p}TD) \otimes (\xi_{i_p}^{\gamma_{i_p}} B)$ are linearly independent vectors, where $\gamma_{i_k} \geq 1$ is the geometric multiplicity of σ for $A_{i_k} + \lambda_{i_k} H$ and $\xi_{i_k}^l (l = 1, \dots, \gamma_{i_k})$ are the left eigenvectors of $A_{i_k} + \lambda_{i_k} H$ corresponding to $\sigma, k = 1, \dots, p$.*

This result generalizes Theorem 8.6 by Hao et al.(Hao et al., 2018) and extend it to a larger class of heterogeneous systems. Also, Theorem 8.8 provides a method to make certain uncontrollable systems controllable by manipulating the components. Numerical examples are provided in **Chapter 4**.

In Chapter 4, heterogeneous systems with distinct state matrices and identical control matrices were considered. However, in **Chapter 5**, we consider heterogeneous networked systems of the form:

$$\dot{x}_i(t) = A_i x_i(t) + \sum_{j=1}^{N} \beta_{ij} H x_j(t) + d_i B_i u_i(t), \quad i = 1, 2, \dots, N \tag{8.12}$$

Then, the above system can be reduced into the compact form

$$\dot{X}(t) = \Omega X(t) + \Psi U(t) \tag{8.13}$$

with,

$$\Omega = \mathcal{A} + L \otimes H \text{ and } \Psi = (D \otimes I)\mathcal{B} \tag{8.14}$$

where $\mathcal{A} = blockdiag\{A_1, A_2, \ldots, A_N\}$ and $\mathcal{B} = blockdiag\{B_1, B_2, \ldots, B_N\}$. Using Theorem 8.7, we derived the following controllability result for the heterogeneous system (8.13)-(8.14).

Theorem 8.9. *(Ajayakumar and George, 2023a) Let T be a non-singular matrix triagularizing matrix L such that $T \otimes I$ commutes with $\mathcal{A}$. If J contains a Jordan block of order $l \geq 2$ corresponding to the eigenvalue λ_{i_0} of L, then assume that $\xi_{ij}^k H = 0$ for all $i = i_0, i_0 + 1, \ldots, i_0 + l - 1, j = 1, 2, \ldots, q_i, k = 1, 2, \ldots, \gamma_{ij}$, where $\xi_{ij}^k, i = 1, 2, \ldots, N, j = 1, 2, \ldots, q_i, k = 1, 2, \ldots, \gamma_{ij}$ are the left eigenvectors of $A_i + \lambda_i H$ corresponding to the eigenvalues $\mu_i^j, i = 1, 2, \ldots, N, j = 1, 2, \ldots, q_i$. Then the networked system* (8.13)-(8.14) *is controllable if and only if*

(i) $e_i T D \neq 0$ *for all* $i = 1, \ldots, N$

(ii) For a fixed i, each left eigenvector ξ of $A_i + \lambda_i H$, $\xi Bj \neq 0$ for some $j \in \{1, 2, \cdots, N\}$ with $[e_i T D]_j \neq 0$; and

(iii) If matrices $A_{i_1} + \lambda_{i_1} H, A_{i_2} + \lambda_{i_2} H, \ldots, A_{i_p} + \lambda_{i_p} H (\lambda_{i_k} \in \sigma(C), k = 1, \ldots, p,$ where $p > 1)$ have a common eigenvalue ρ, then $(e_{i_1} T D \otimes \xi_{i_1}^1)\mathcal{B}, \cdots, (e_{i_1} T D \otimes \xi_{i_1}^{\gamma_{i_1}})\mathcal{B}, \ldots, (e_{i_p} T D \otimes \xi_{i_p}^1)\mathcal{B}, \ldots, (e_{i_p} T D \otimes \xi_{i_p}^{\gamma_{i_p}})\mathcal{B}$ are linearly independent vectors, where $\gamma_{i_k} \geq 1$ is the geometric multiplicity of the eigenvalue ρ for the matrix $A_{i_k} + \lambda_{i_k} H$ and $\xi_{i_k}^l (l = 1, \ldots, \gamma_{i_k})$ are the left eigenvectors of $A_{i_k} + \lambda_{i_k} H$ corresponding to $\rho, k = 1, \ldots, p$.

This result generalizes Theorem 8.6 and Theorem 8.8 and provide a more general framework which establishes the impact of individual node dynamics, network topology, inner coupling matrices, etc., on the controllability of networked systems .

Along with Theorem 8.9, we derived the following sufficient condition for a general heterogeneous networked system in **Chapter 5** where the inner-coupling matrices are also distinct in each node and the network topology matrix is triangular.

Theorem 8.10. *(Ajayakumar and George, 2023a) Assume that L is an upper triangular matrix. Let $\sigma(A_i + \beta_{ii} H_i) = \{\mu_i^1, \ldots, \mu_i^{q_i}\}$ be the set of eigenvalues of $A_i + c_{ii} H_i, i = 1, 2, \ldots, N$. Then the set of all eigenvalues of Ω is given by $\sigma(\Omega) = \{\mu_1^1, \mu_1^2, \ldots, \mu_1^{q_1}, \ldots, \mu_N^1, \mu_N^2, \ldots, \mu_N^{q_N}\}$. Let $\xi_{ij}^k, k = 1, 2, \ldots, \gamma_{ij}$ be the left eigenvectors of $A_i + \beta_{ii} H_i$ associated with the eigenvalue μ_i^j, where γ_{ij} is the geometric multiplicity of the eigenvalue μ_i^j for the matrix $A_i + \beta_{ii} H_i$. If $\xi_{ij}^k H_i = 0$, for $i = 1, 2, \ldots, N-1, j = 1, 2, \ldots, q_i, k = 1, 2, \ldots, \gamma_{ij}$, then $e_i \otimes \xi_{ij}^1, e_i \otimes \xi_{ij}^1, \ldots e_i \otimes \xi_{ij}^{\gamma_{ij}}$, are the left eigenvectors of Ω associated with the eigenvalues μ_i^j.*

In **Chapter 6**, we considered the controllability problem of networked systems with individual nodes having both linear and non-linear components. The individual node dynamics is given by:

$$\dot{x_i}(t) = A_i x_i(t) + \sum_{j=1}^{N} \beta_{ij} H x_j(t) + d_i B_i u_i(t) + f_i(t, x_i(t)), \quad i = 1, 2, \ldots, N \tag{8.15}$$

The non-linear system can be rewritten in a compact form as

$$\dot{X}(t) = \Omega X(t) + \Psi u(t) + F(t, X(t)) \tag{8.16}$$

with

$$\Omega = A + L \otimes H, \; \Psi = blockdiag\{d_1 B_1, \ldots d_N B_N\}$$

and

$$F(t, x(t)) = \left[f_1(t, x_1(t))^T, \ldots, f_N(t, x_N(t))^T\right)$$

where $A = blockdiag\{A_1, \ldots, A_N\}$. The linear part of the networked system was assumed to be controllable and non-linear components were assumed to satisfy Lipschitz condition. We derived a controllability condition for such systems by employing *Generalized Banach Contraction Principle*.

Chapter 7 explores the notion of generic controllability in networked systems, which is a type of structural controllability. Here the individual node dynamics is given by

$$\begin{aligned} \dot{x}_i(t) &= A_i x_i(t) + B_i w_i(t) \\ y_i(t) &= C_i x_i(t) \end{aligned} \tag{8.17}$$

where the input signal for the node i is given by the weighted combination of control signals in line with the network graph

$$w_i(t) = \sum_{i=1}^{N} \beta_{ij} y_j(t) + \sum_{l=1}^{m} \delta_{il} u_l(t) \tag{8.18}$$

Then the compact form of $\sum_{\mathcal{N}}$ is given by

$$\sum_{\mathcal{N}} : \dot{x}(t) = \Omega x(t) + \Psi u(t) \tag{8.19}$$

where $x(t) = (x_1(t), \ldots, x_m(t))^T$ and $u(t) = (u_1(t), \ldots, u_m(t))^T$. The matrices Ω and Ψ representing the state and control matrices of $\sum_{\mathcal{N}}$, respectively, have dimensions $Nn \times Nn$ and $Nn \times m$. They are of the following form:

$$\Omega = \begin{bmatrix} A_1 + \beta_{11} B_1 C_1 & \beta_{12} B_1 C_2 & \ldots & \beta_{1N} B_1 C_N \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{N1} B_N C_1 & \beta_{N2} B_N C_2 & \ldots & A_N + \beta_{NN} B_N C_N \end{bmatrix}$$

and

$$\Psi = \begin{bmatrix} \delta_{11}B_1 & \delta_{12}B_1 & \dots & \delta_{1m}B_1 \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{N1}B_N & \delta_{N2}B_N & \dots & \delta_{Nm}B_N \end{bmatrix}$$

In this model, the matrices $(A_i, B_i, C_i)'s$ are assumed to be exact and known, but the network communication strength are not fixed precisely. That is, we know whether the entries are zero or non-zero, but we do not know the exact parameter values. Commault et al. (Commault and Kibangou, 2019) give the following set of conditions which are necessary and sufficient for the generic controllability of interconnected systems with identical dynamical nodes.

Theorem 8.11. *(Commault and Kibangou, 2019) Consider a network $\mathcal{N}$ with N internal nodes, m control nodes with $N > m$, and its graph $G(\mathcal{N})$. Assume that all nodes are identical, SISO, n^{th}-order dynamical systems defined by matrices A, B, C. The global system $\Sigma_{\mathcal{N}}$ is generically controllable if and only if the following conditions hold:*

(i) *The pair (A, B) is controllable.*

(ii) *The pair (C, A) is observable.*

(iii) *The graph $G(\mathcal{N})$ is control-connected.*

(iv) *The internal nodes of $G(\mathcal{N})$ can be covered by a disjoint set of stems and cycles.*

In **Chapter 7**, we show that the conditions $(i), (ii)$ and (iii) of Theorem 8.11 are necessary for the heterogeneous networked system (8.17)-(8.19).

Apart from the results reported in the thesis, there are a few interesting and challenging research problems that necessitate additional investigation. They are summarized as follows. The controllability analysis for a general heterogeneous networked system where the individual nodes have distinct node dimensions has not been explored much in the existing literature. We have obtained some necessary conditions for controllability of general heterogeneous networked systems in Thomas et al.(Thomas et al., 2023) and we are currently exploring this problem further. We have also established a necessary and sufficient condition for observability of networked systems in this work. Investigating the controllability of networked systems with delays in both state and control matrices also presents a compelling research area for future research. In the thesis, all the models considered are LTI systems. Controllability of linear time variant networked systems is another key area of research that we intend to explore in near future.

List of Publications

1. Ajayakumar, A., & George, R. K. (June 2023). A Note on Controllability of Directed Networked System with Heterogeneous Dynamics, *IEEE Transactions on Control of Network Systems*, vol. 10, no. 2, pp. 575-578.

2. Ajayakumar, A., & George, R. K. (2023). Controllability of networked systems with heterogeneous dynamics. *Math. Control Signals Syst.* 35, 307-326.

3. Ajayakumar, A., & George, R. K. (2023). Controllability of a Class of Heterogeneous Networked Systems. *Foundations*, 3(2), 167-180.

4. Ajayakumar, A., & George, R. K. (2023). Controllability of networked systems with nonlinearities. *Indain Journal of Mathematics*, 65(2), 267-277.

5. Ajayakumar, A., & George, R. K. (2022). Controllability of Linear Time Invariant Networked Systems: A Review. *chetana: An Ivanian Journal for Scientific Research*, 1(1), 26-34.

6. Ajayakumar, A., & George, R. K. (2023). A Note on Generic Controllability of Networks with Identical SISO Dynamical Nodes. *Journal of Mathematical Control Science and Applications*, 9(2), 1-8.

List of Papers Presented

1. 'A Note on Generic Controllability Networks with Identical SISO Dynamical Nodes', *International Conference on Differential Equations and Control Problems*, Organized by School of Mathematical and Statistical Sciences, IIT Mandi, June 2023.

2. 'Controllability of LTI Networked Systems', *International e-Conference on Number Theory and Differential Equations*, Organized by Central University, Karnataka, December 2021.

3. 'Controllability of LTI Networked Systems with Heterogeneous Dynamics', *Symposium on Differential Equations; Analysis, Computation and Applications*, Organized by IIT, Roorkee, December 2021.

4. ‘Generic Controllability of Networked Systems with Heterogeneous Dynamics’, *International Conference on Recent Advances in Pure and Applied Algebra*, Organized by International Academy of Physical Sciences and NIT, Jamshedpur, October 2021.

5. ‘Controllability of Networked Control Systems with Triangular Network Topology’, *International Virtual Conference on Mathematical Modelling, Analysis and Computing*, Organized by Thiruvalluvar University, July 2021.

6. ‘Controllability of LTI Networked Systems with Heterogeneous Dynamics’, *National Conference on Mathematical Control Theory*, Organized by NIT, Puducherry and IIST, Thiruvananthapuram, December 2020.

List of Conferences Attended

1. National Conference on Applied Mathematics and Numerics, Organized by Department of Mathematics, Mar Ivanios College, Thiruvananthapuram and IIST, Thiruvananthapuram, March 2022.

2. National Seminar on Differential Equations and its Applications, Organized by Department of Mathematics, University College, Thiruvananthapuram, November 2019.

3. National Conference on Stochastic Differential Equations and Applications, Organized by IIST, Thiruvananthapuram and IIT, Roorkee, June 2019.

Awards

- Best Oral Presentation Award for ‘A Note on Generic Controllability Networks with Identical SISO Dynamical Nodes’, presented at the *International Conference on Differential Equations and Control Problems*, Organized by School of Mathematical and Statistical Sciences, IIT Mandi, June 2023.

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